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NOTE

A NOTE ON THE EXISTENCE OF CONTINUOUS FUNCTIONALS

J. Lawrence CARTER
Computer Science Department, IBM Watson Research Center, Yorktown Heights, NY 10598, U.S.A.

Ronald FAGIN
Computer Science Department, IBM Research Laboratory, San Jose, CA 95193, U.S.A.

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Abstract. Let $P=\{p_i \mid i\in I\}$ and $Q=\{q_i \mid i\in I\}$ be sets of partial functions with the same index set I. We say that Φ is an interpolating function (from P to Q) if $\Phi(p_i)=q_i$ for each i. We give simple necessary and sufficient conditions for the existence of a monotone interpolating functional. We show that these same conditions are necessary and sufficient for the existence of a continuous interpolating functional if the index set I is finite, but that they are not sufficient if the index set is infinite.

1. Introduction

Recently, there has been interest in languages for functional programming (see [1, 2]). In these languages, there are two basic entities:

(a) partial functions and

(b) functionals (which map partial functions into partial functions).

Monotone functionals and continuous functionals play a fundamental role in this research. We therefore consider basic foundational questions as to their existence.

2. Definitions

A *partial function* from A to B is a function from a subset of A to B. In this paper, we will deal with the set $\mathcal{F}$ of partial functions from N to N, where N is the set $\{0, 1, 2, \ldots\}$ of natural numbers. If f is a partial function and if a is a natural number not in the domain of f, then we write $f(a)=\bot$, and we say that $f(a)$ is *undefined*, or that f is *undefined on a*.

If a and b are elements of $N\cup\{\bot\}$, then we say $a\sqsubseteq b$ if either $a=b$ or $a=\bot$. If f and g are partial functions, then we say $f\sqsubseteq g$ if $f(x)\sqsubseteq g(x)$ for all x in N. Thus,

$f \sqsubseteq g$ if whenever $f(x)$ is defined, then $f(x) = g(x)$. Intuitively, $f \sqsubseteq g$ means that f is the restriction of g to a smaller (or equal) domain.

We say that f and g are *compatible* if whenever $f(x)$ and $g(x)$ are both defined, then $f(x) = g(x)$. Let $S = \{f_i \mid i \in I\}$ be a pairwise compatible set of partial functions (that is, f_i and f_j are compatible for each i and j). Then sup S is defined to be the partial function whose value on x is $\perp$ when $f_i(x)$ is undefined for each i, and $f_i(x)$ when $f_i(x)$ is defined for some i. Note that sup S is well-defined, since if $f_i(x)$ and $f_j(x)$ are both defined, then the values are the same. The partial function sup S is the smallest partial function such that $f \sqsubseteq \sup S$ for each f in S. Notice that if $f_1 \sqsubseteq f_2 \sqsubseteq \ldots$, then $\{f_i\}$ is pairwise compatible. We write $\lim_{i\to\infty} f_i = g$ to mean that $f_1 \sqsubseteq f_2 \sqsubseteq \cdots$ and $\sup\{f_i\} = g$.

A *functional* is a mapping from $\mathcal{F}$ to $\mathcal{F}$. There are several types of functionals that are of particular interest. A functional Φ is *monotone* if it preserves $\sqsubseteq$, that is, if whenever $f \sqsubseteq g$, then $\Phi(f) \sqsubseteq \Phi(g)$. A functional is *continuous* if it is monotone and preserves limits, i.e., if $\lim_{i\to\infty} p_i = p$, then $\lim_{i\to\infty} (\Phi(p_i)) = \Phi(p)$.

Let $P = \{p_i \mid i \in I\}$ and $Q = \{q_i \mid i \in I\}$ be sets of partial functions. We say that Φ is an *interpolating functional from P to Q* (or simply an *interpolating functional*) if $\Phi(p_i) = q_i$ for each i. We will consider the existence of monotone or continuous interpolating functionals.

3. Interpolating functionals

Theorem 1. *Let $\{p_i \mid i \in I\}$ and $\{q_i \mid i \in I\}$ be sets of partial functions with the same index set I. Then there exists a monotone interpolating functional Φ if and only if*:

(i) *whenever $p_i \sqsubseteq p_j$, then $q_i \sqsubseteq q_j$, and*

(ii) *if p_i and p_j are compatible, then so are q_i and q_j.*

Proof. Assume first that a monotone interpolating functional Φ exists. Then condition (i) is clearly necessary by the definition of monotonicity. Condition (ii) is also necessary, because if p_i and p_j are compatible, then there exists p such that $p_i \sqsubseteq p$ and $p_j \sqsubseteq p$; then $q_i = \Phi(p_i) \sqsubseteq \Phi(p)$ and similarly $q_j \sqsubseteq \Phi(p)$, and so q_i and q_j are compatible.

Suppose now that conditions (i) and (ii) hold; we will show that a monotone interpolating functional Φ exists. Definine $\Phi : \mathcal{F} \to \mathcal{F}$ by $\Phi(p) = \sup\{q_i \mid p_i \sqsubseteq p\}$. (The sup over the empty set is the partial function whose domain is empty, that is, the partial function that is undefined everywhere.)

We must first establish that this definition is legal. The set $\{p_i \mid p_i \sqsubseteq p\}$ of partial functions is pairwise compatible, since each element of this set is $\sqsubseteq p$. Condition (ii) then implies that $\{q_i \mid p_i \sqsubseteq p\}$ is pairwise compatible; thus, Φ is well-defined.

To show that Φ is monotone, we suppose that $r_1 \sqsubseteq r_2$. Then $\{i \mid p_i \sqsubseteq r_1\} \subseteq \{i \mid p_i \sqsubseteq r_2\}$. Therefore, $\{q_i \mid p_i \sqsubseteq r_1\} \subseteq \{q_i \mid p_i \sqsubseteq r_2\}$. It follows that $\Phi(r_1) \sqsubseteq \Phi(r_2)$, as desired.

Finally, we must show that $\Phi(p_k) = q_k$ for all k in I. If $p_i \sqsubseteq p_k$, then by (i), $q_i \sqsubseteq q_k$.

Thus $\Phi(p_k)$ is the sup over a collection of objects that contains the maximal element q_k, so $\Phi(p_k) = q_k$.

We will give an example later which shows that the monotone functional defined in the above proof is not necessarily continuous, even if the index set I is finite.

We now show that when the index set I is finite, then the conditions of Theorem 1 are necessary and sufficient for the existence of a *continuous* interpolating functional, although this functional may not be the one constructed in the above proof.

Theorem 2. *Suppose that* $\{p_i \mid i = 1, \ldots, n\}$ *and* $\{q_i \mid i = 1, \ldots, n\}$ *are finite sets of partial functions in* $\mathcal{F}$. *Then conditions* (i) *and* (ii) *of Theorem 1 are necessary and sufficient conditions for the existence of a continuous interpolating functional.*

Proof. The necessity of (i) and (ii) follows immediately from Theorem 1, since if a continuous functional exists, it must be monotone. However, as we noted, the straightforward construction of Φ used in Theorem 1 does not necessarily yield a continuous functional, so we must work a little harder to show the sufficiency of (i) and (ii).

Suppose that conditions (i) and (ii) hold. For each j and k between 1 and n for which it is false that $p_j \sqsubseteq p_k$, choose x_{jk} as follows:

If p_j and p_k are not compatible, then pick x_{jk} such that p_j and p_k are both defined but unequal on x_{jk}. Such an x_{jk} is certain to exist by the definition of compatible.

Otherwise, pick x_{jk} so that $p_j(x)$ is defined and $p_k(x)$ is not. Since p_j and p_k are compatible but it is false that $p_j \sqsubseteq p_k$, such a point x_{jk} must exist.

Intuitively, x_{jk} is a witness to the fact that p_j and p_k are incompatible, or that $p_j \sqsubseteq p_k$ does not hold. Let D be the set of all witnesses x_{jk} chosen in this way. Then D is a finite set, since there are only a finite number of p_i's. Definite p'_i to be the restriction of p_i to D, that is, $p'_i(x) = p_i(x)$ if $x \in D$, and $p'_i(x) = \bot$ otherwise. Finally, define the functional $\Phi(r) = \sup\{q_i \mid p'_i \sqsubseteq r\}$. We now show that Φ has the necessary properties to be an interpolating functional.

Φ is well-defined. We need only to show that $\{q_i \mid p'_i \sqsubseteq r\}$ is a compatible set. Suppose that for some r, this set contained two elements, say q_j and q_k, that were incompatible. By property (ii), p_j and p_k must be incompatible. Hence, D must contain a witness x_{jk} such that that p_j and p_k are defined and unequal on x_{jk}. But then it is impossible that both $p'_j \sqsubseteq r$ and $p'_k \sqsubseteq r$, which is a contradiction.

Φ is monotone. If $r_1 \sqsubseteq r_2$, then $\{i \mid p'_i \sqsubseteq r_1\} \subseteq \{i \mid p'_i \sqsubseteq r_2\}$, and thus $\Phi(r_1) \sqsubseteq \Phi(r_2)$.

Φ is continuous. This is where the finiteness of the number of p_i's (and hence of D) is used. Suppose $r_1 \sqsubseteq r_2 \sqsubseteq \cdots$ and $\lim_{i\to\infty} r_i = r$. For each x in D, there is some integer m_x s.t. $r_i(x) = r(x)$ for all i with $i \geqslant m_x$. (This is true even if $r(x) = \bot$, since in this case $r_i(x) = \bot$ for all i.) Since D is finite, we can set $m = \max\{m_x \mid x \in D\}$. Then r and r_i agree on all points of D for $i \geqslant m$; hence, $\Phi(r_i)$ must be $\Phi(r)$ if $i \geqslant m$.

For each i, we know that $\Phi(r_i) \sqsubseteq \Phi(r_{i+1})$ by the monotonicity of Φ. Thus $\lim_{i\to\infty} \Phi(r_i)$ exists and equals $\Phi(r)$.

$\Phi(p_k) = q_k$ *for* $k = 1, 2, \ldots, n$. Since $p'_k \sqsubseteq p_k$, it follows that $q_k \in \{q_i \mid p'_i \sqsubseteq p_k\}$. So, $q_k \sqsubseteq \sup\{q_i \mid p'_i \sqsubseteq p_k\} = \Phi(p_k)$. Before showing the reverse inequality we make the observation that if $p'_j \sqsubseteq p_k$, then in constructing D it was impossible to find a witness x_{jk} such that $p_j(x_{jk}) \sqsubseteq p_k(x_{jk})$ fails; hence it follows that $p_j \sqsubseteq p_k$.

Now,

$$
\begin{aligned}
\Phi(p_k) &= \sup\{q_j \mid p'_j \sqsubseteq p_k\} && \text{(by definition)}\\
&\sqsubseteq \sup\{q_j \mid p_j \sqsubseteq p_k\} && \text{(by the above observation)}\\
&\sqsubseteq \sup\{q_j \mid q_j \sqsubseteq q_k\} && \text{(by property (i))}\\
&= q_k.
\end{aligned}
$$

We have shown that $\Phi(p_k) \sqsubseteq q_k$ and that $q_k \sqsubseteq \Phi(p_k)$. Hence $\Phi(p_k) = q_k$.

When each of the p_i's are total functions (that is, when each has domain $\mathbf{N}$), then there is a particularly simple condition that characterizes when there is a continuous interpolating functional.

Theorem 3. *Suppose that* $\{p_i \mid i = 1, \ldots, n\}$ *is a finite set of total functions in* $\mathcal{F}$, *and that* $\{q_i \mid i = 1, \ldots, n\}$ *is a finite set of partial functions in* $\mathcal{F}$. *Then a necessary and sufficient condition for the existence of a continuous interpolating functional is that whenever* $p_i = p_j$, *then* $q_i = q_j$.

Proof. Since p_i is total, there is no p in $\mathcal{F}$ other than p_i itself such that $p_i \sqsubseteq p$. Thus, conditions (i) and (ii) are each equivalent to the condition that $p_i = p_j$ implies $q_i = q_j$. The result then follows immediately from Theorem 1.

4. Counterexamples

We first show that the construction of Theorem 1 (which produces a monotone functional) does not necessarily produce a continuous functional, even if the index set I is finite and if the p_i's and q_i's are total. Let p_0 and q_0 be the constant function that is identically 0. Let $\{p_i \mid i \in I\}$ and $\{q_i \mid i \in I\}$ of Theorem 1 be the singleton sets $\{p_0\}$ and $\{q_0\}$, respectively. It is easy to verify that in this case, the functional Φ as defined in the proof of Theorem 1 maps p_0 into q_0, and maps every other partial function p into the partial function Ω that is undefined everywhere. We now show that Φ is not continuous. For $i = 1, 2, \ldots$, let r_i be the partial function with $r_i(x) = 0$ when $x \leq i$ and with $r_i(x) = \perp$ otherwise. Then $\lim_{i\to\infty} r_i = p_0$, and so $\Phi(\lim_{i\to\infty} r_i) = \Phi(p_0) = q_0$. However, $\lim_{i\to\infty} \Phi(r_i) = \lim_{i\to\infty} \Omega = \Omega$. Hence, $\Phi(\lim_{i\to\infty} r_i) \neq \lim_{i\to\infty} \Phi(r_i)$, which shows that Φ is not continuous.

We now show that if the index set I is infinite, then conditions (i) and (ii) of Theorems 1 and 2 are not sufficient to guarantee the existence of a continuous interpolating functional.

For each integer $n \geq 0$, let p_n be the characteristic function of n, that is, $p_n(n) = 1$ and $p_n(i) = 0$ for $i \neq n$. Let p_0 and q_0 be the constant function that is identically 0, and for $i > 0$ let q_i be the constant function that is identically 1. We now show that there can be no continuous functional Φ such that $\Phi(p_i) = q_i$ for each $i \geq 0$. Assume that there were a continuous interpolating functional Φ. As before, let r_i be the partial function with $r_i(x) = 0$ if $x \leq i$, and with $r_i(x) = \perp$ otherwise. We will demonstrate that $\Phi(r_k)$ must be Ω, the totally undefined partial function, for each k. This leads to a contradiction, however, since we would then have $q_0 = \Phi(p_0) = \Phi(\lim_{i\to\infty} r_i) = \lim_{i\to\infty}\Phi(r_i) = \lim_{i\to\infty}\Omega = \Omega$, but the leftmost term is q_0, the constant zero function.

We now substantiate the claim that $\Phi(r_k) = \Omega$ for each k. First notice that $r_k \sqsubseteq p_{k+1}$; so, by monotonicity of Φ, it follows that $\Phi(r_k) \sqsubseteq \Phi(p_{k+1}) = q_{k+1}$, the constant one function. On the other hand, $r_k \sqsubseteq p_0$, so by monotonicity of Φ, it follows that $\Phi(r_k) \sqsubseteq \Phi(p_0) = q_0$, the constant zero function. The only way it is possible that $\Phi(r_k)$ is less than or equal to both the constant zero function and the constant one function is that $\Phi(r_k) = \Omega$, which was to be shown.

5. Related Work

There is a natural generalization of the ideas presented above: Instead of dealing with partial functions from **N** to **N**, one can investigate (total) functions from A to B, where B is a set partially ordered by $\leq$. The definition of $\sqsubseteq$ for functions from A to B is simply: $f \sqsubseteq g$ iff $f(x) \leq g(x)$ for all $x \in A$. To fit our case into this generalization, we would let $A = \mathbf{N}$ and $B = \mathbf{N} \cup \{\perp\}$ with the partial order on B given by $x \leq y$ if $x \sqsubseteq y$ (as defined above). The reader interested in constructing continuous functionals interpolating such functions should consult [3] for some relevant theory.

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