Appeared in:
Proc. 17th ACM SIGACT Symp. on Theory of Computing, Providence, May 1985, pp. 305-315.

An Internal Semantics for Modal Logic: Preliminary Report

Ronald Fagin
IBM Research Laboratory, 5600 Cottle Road, San Jose, CA 95193

Moshe Y. Vardi
CSLI, Ventura Hall, Stanford University, Stanford, CA 94305

Abstract: In Kripke semantics for modal logic, "possible worlds" and the possibility relation are both primitive notions. This has both technical and conceptual shortcomings. From a technical point of view, the mathematics associated with Kripke semantics is often quite complicated. From a conceptual point of view, it is not clear how to use Kripke structures to model knowledge and belief, where one wants a clearer understanding of the notions that are primitive in Kripke semantics. We introduce *modal structures* as models for modal logic. We use the idea of possible worlds, but by directly describing the "internal semantics" of each possible world. It is much easier to study the standard logical questions, such as completeness, decidability, and compactness, using modal structures. Furthermore, modal structures offer a much more intuitive approach to modelling knowledge and belief.

1. Introduction

Modal logic can be described briefly as the logic of necessity and possibility, of "must be" and "may be". (One should not take "necessity" and "possibility" literally. "Necessarily" can mean "according to the laws of physics" or "according to my beliefs", or even "after the program terminates".) Modal logic was discussed by several authors in ancient times, notably Aristotle in *De Interpretatione* and *Prior Analystics*, and by medieval logicians, but like most work before the modern period, it was non-symbolic, and not particularly systematic in approach. The first *symbolic* and *systematic* approach to the subject appears to be the work of Lewis beginning in 1912 and culminating in the book *Symbolic Logic* with Langford in 1932. Since then modal logic has been extensively studied by logicians and philosophers ([Ch] is a good textbook). More recently, modal logic has been applied in several areas of computer science, such as artificial intelligence [MH], program verification and synthesis [MW,Pn,Pra1], hardware specification [Bo,RS], protocol specification and verification [CES,SM], database theory [CCF,Li], and distributed computing [HM1].

Lewis' semantics for modal logic was of an algebraic cast. Algebraic semantics, however, though technically adequate (cf. [Gu,Mc,McT,Ts]), is nevertheless not very intuitive. In 1946 Carnap [Ca1,Ca2] suggested using the more intuitive approach of *possible worlds* to assign semantics to modalities. According to this approach, one starts with a set of possible worlds. Then statements of the form $\Box p$ (i.e., *p is necessarily true*) are interpreted in the following way: $\Box p$ is true if p is true in every "possible world". (The idea that necessity is truth in all possible worlds is often credited to Leibniz [Bat], though this is historically debatable.) Possible-worlds semantics was further developed independently by several researchers [Bay, Hi1, Hi2, Ka, Kr1, Me, Mo1, Pri], reaching its current form with Kripke [Kr2]. The basic idea of the development is to consider, instead of a set of worlds that are possible outright, a set of worlds that are, or are not, *possible with respect to each other*. *Kripke structures* capture this intuitive idea. A Kripke structure can be viewed as a labeled directed graph: the nodes are the possible worlds labeled by truth assignments, and a world v is possible with respect to a world u if there is an edge from u to v.

Kripke structures were immensely successful mathematical tools and served as the basis for extremely fertile research. Nevertheless, they do suffer from both technical and conceptual shortcomings. From a technical point of view, the mathematics associated with them is often quite complicated. For example, completeness proofs are either non-elegant

[Kr2] (Kripke himself described his completeness proof as "rather messy"), or non-constructive [Ma,Lem2]. (Only in 1975 did Fine come with an elegant and constructive proof [Fi], but his proof is far from straightforward.) Also, the standard technique for proving decidability is to show that the logic has the *finite model property*, which is not straightforward at all. Furthermore, in order to model different modal logics, certain graph-theoretic contraints on the possibility relation between possible worlds have to be imposed; these constraints are very often far from intuitive, and sometimes they are not even first-order definable [Go].

From a conceptual point of view, it is not clear that Kripke structures are as intuitive as they are supposed to be. The basic problem is that in Kripke semantics the notion of a possible world is a *primitive* notion. (Indeed possible worlds are called *reference points* in [Mo2] and *indices* in [Sc].) This works well in applications where it is intuitively clear what a possible world is. For example, in *dynamic logic* a possible world is just a program state [Pra1], i.e., an assignment of values to the variables and to the location counter, and in *temporal logic* a possible world is just a point in time [Bu]. But in applications where it is not clear what a possible world is, e.g., in *epistemic logic*, the logic of knowledge and belief, how can we construct a Kripke structure without understanding its basic constituents? Indeed, in dynamic and temporal logic one constructs first the structures, and then proceeds to find the axioms [Bu,KP], while in epistemic logic one first selects axioms and then tailors the structures to the axioms [HM2,Re].

Furthermore, if we want to *apply* modal logic it is often necessary to construct models for particular situations. But if we have no means of explicitly describing the possible worlds, how can we construct Kripke structures to model particular situations? Indeed, as pointed out in [FHV], there are simple scenarios in distributed environments that cannot be easily modelled by Kripke structures.

We believe that our approach, which is to describe explicitly the "internal" semantics of a possible world, is a much more intuitive approach to modelling. We introduce *modal structures* as models for modal logic. We use the idea of possible worlds, but in Carnap's style rather than Kripke's style. Thus, we define a modal structure to be essentially a set of modal structures. This is of course a circular definition, and to make it meaningful, we define *worlds* inductively, by constructing worlds of greater and greater *depth*. A world of depth 0 is a description of reality, i.e., a truth assignment; a world of depth 1 is essentially a set of worlds of depth 0; a world of depth 2 is essentially a set of worlds of depth 1; etc. Modal structures are worlds of depth ω, and their recursive structure enables us to assign meaning to iterated modalities.

Having introduced modal structures, we investigate their relationship to Kripke structures. It turns out that modal structures model individual nodes in Kripke structures, while Kripke structures model collections of modal structures. Thus, modal structures can be seen as duals to Kripke structures. Nevertheless, it is much easier to study the standard logical questions, such as completeness, decidability, and compactness, using modal structures. The crucial point is that satisfaction of a formula in a modal structure depends only on a certain *finite* part of that structure. Furthermore, the "size" of that part depends on the "size" of the formula. Thus, the proofs of decidability and compactness are almost straightforward, and the completeness proof is both elegant and constructive. We urge the reader to compare our proofs to previous proofs (e.g., [Fr, Kr2, Lem2, Ma, Mc]) in order to appreciate their elegance.

Beyond the technical usefulness of modal structures, we claim that they are more intuitive and more appropriate to conceptual modelling. For example, the simple scenarios in distributed environments mentioned above can be modelled by modal structures in a straightforward way [FHV]. We also demonstrate the intuitiveness of modal structures by modelling belief and by modelling joint knowledge, then using our techniques to prove decidability and completeness in these cases (and compactness in the case of belief; compactness fails for joint knowledge).

The double perspective that we have now on modal logic, namely, Kripke structures and modal structures, turns out to be very useful in proving optimal upper bounds for the complexity of the decision problem. By their graph-theoretic nature, Kripke structures are amenable to automata-theoretic techniques [ES,Str,VW]. By combining our results for modal structures with a new automata-theoretic technique for Kripke structures, we prove that several of the logics that we study are complete in PSPACE.

2. Modal structures

Basic definitions. We now define structures that capture the essence of the "possible worlds" approach. (For the sake of simplicity, we restrict ourselves here to *normal* modal logics [Ch]. Nevertheless, this is not an inherent limitation of our approach.) In anticipation of subsequent developments where modalities correspond to "attitudes" of agents or players, we allow multiple modalities

$\Box_1, \ldots, \Box_n$. A good way to interpret the statement $\Box_i p$ is "after program i terminates, p must be true". That is, one can view the structures we define here as models for the modal logic of atomic programs.

We assume a fixed finite set of primitive propositions, and a fixed finite index set $\mathcal{I}$ such that for each $i \in \mathcal{I}$, there is a modality $\Box_i$. Intuitively, a modal structure has various "levels", where the 0th level is a truth assignment to the primitive propositions, and where the kth level contains a set of "possible k-ary worlds" for each modality. Formally, we define a *0th-order assignment*, f_0, to be a truth assignment to the primitive propositions. We call $<f_0>$ a *1-ary world* (since its "length" is 1). Assume inductively that k-ary worlds $<f_0, \ldots, f_{k-1}>$ have been defined. Let W_k be the set of all k-ary worlds. A *kth-order assignment* is a function f_k: $\mathcal{I} \to 2^{W_k}$. Intuitively, f_k associates with each modality a set of "possible k-ary worlds". There is a semantic restriction on f_k, which we shall discuss shortly. We call $<f_0, \ldots, f_k>$ a *$(k+1)$-ary world.* An infinite sequence $<f_0 f_1 f_2, \ldots>$ is called a *modal structure* if the *prefix* $<f_0, \ldots, f_{k-1}>$ is a k-ary world for each k.

The semantic restriction on worlds $<f_0, \ldots, f_k>$ that we mentioned earlier, is: $<g_0, \ldots g_{k-2}> \in f_{k-1}(i)$ iff there is a $(k-1)$st-order assignment g_{k-1} such that $<g_0, \ldots, g_{k-2} g_{k-1}> \in f_k(i)$, if $k > 1$. This restriction says that the set of $(k-1)$-ary worlds associated with modality i are prefixes of the set of k-ary worlds associated with modality i. It is straightforward to verify that this "compatibility" between f_{k-1} and f_k also holds between f_j and f_k if $0 < j < k$. Thus, the set of k-ary worlds associated with modality i (namely, $f_k(i)$) determines the set of j-ary worlds associated with modality i (namely, $f_j(i)$) if $0 < j < k$. Later on, when we use modal structures to model propositional attitudes (such as knowledge and belief), we have to impose further semantic restrictions.

The following lemma is obvious, but crucial.

Lemma 2.1. *There are only a finite number of k-ary worlds, for each k.*

Because of our semantic restriction, it may not be obvious to the reader that each world $<f_0, \ldots f_{k-1}>$ is the prefix of a modal structure $<f_0, \ldots f_{k-1} f_k, \ldots>$. We now show that this is the case. We note that later, when we introduce logics where the modal structures have further semantic restrictions, it is even less clear whether each world is the prefix of a modal structure.

Theorem 2.2. *Each world is the prefix of a modal structure.*

Proof. Let $<f_0, \ldots f_{k-1}>$ be a world. Define $f_j(i) = \{ <g_0, \ldots, g_{j-1}> : <g_0, \ldots, g_{j-1}>$ is a j-ary world and $<g_0, \ldots g_{k-2}> \in f_{k-1}(i)\}$ for each $j \geq k$ and each modality i. Thus, $f_j(i)$ contains every possible j-ary world that has a member of $f_{k-1}(i)$ as a prefix. It can be verified that $<f_0, \ldots f_{k-1} f_k, \ldots>$ is a modal structure. ∎

Syntax and semantics. The set of *formulas* is the smallest set that contains the primitive propositions, is closed under Boolean connectives and contains $\Box_i \varphi$ if it contains φ. The *depth* of a formula φ is the depth of nesting of the $\Box_i$'s in φ.

We are almost ready to define what it means for an modal structure to *satisfy* a formula. We begin by defining what it means for an $(r+1)$-ary world $<f_0, \ldots f_r>$ to satisfy formula φ, written $<f_0, \ldots f_r> \models \varphi$, if $r \geq \text{depth}(\varphi)$.

1. $<f_0, \ldots f_r> \models p$, where p is a primitive proposition, if p is true under the truth assignment f_0.
2. $<f_0, \ldots f_r> \models \sim\varphi$ if $<f_0, \ldots f_r> \not\models \varphi$.
3. $<f_0, \ldots f_r> \models \varphi_1 \wedge \varphi_2$ if $<f_0, \ldots f_r> \models \varphi_1$ and $<f_0, \ldots f_r> \models \varphi_2$.
4. $<f_0, \ldots f_r> \models \Box_i \varphi$ if $<g_0, \ldots g_{r-1}> \models \varphi$ for each $<g_0, \ldots g_{r-1}> \in f_r(i)$.

Lemma 2.3. *Assume that* $\text{depth}(\varphi) = k$ *and* $r \geq k$. *Then* $<f_0, \ldots f_r> \models \varphi$ *iff* $<f_0, \ldots f_k> \models \varphi$.

We say that the modal structure $f = <f_0, f_1, \ldots>$ satisfies φ, written $f \models \varphi$, if $<f_0, \ldots f_k> \models \varphi$, where $k = \text{depth}(\varphi)$. This is a reasonable definition, since if $w = <f_0, \ldots, f_r>$ is an arbitrary prefix of f such that $r \geq k$, then it then follows from Lemma 2.3 that $f \models \varphi$ iff $w \models \varphi$.

Relationship to Kripke structures. A Kripke structure M is a triple $(S, \pi, \mathcal{R})$, where S is a set of *states*, $\pi(s)$ is a truth assignment to the primitive propositions for each state $s \in S$, and $\mathcal{R}(i)$ is a binary relation on S for each modality $\Box_i$. Intuitively, $(s,t) \in \mathcal{R}(i)$ iff starting in world s, the world t is possible, according to modality $\Box_i$. In terms of our interpretation where modalities correspond to programs, $(s,t) \in \mathcal{R}(i)$ iff starting in state s and running program i, it is possible to terminate in state t. We now define what it means for a formula φ to be *satisfied* at a state s of M, written $M, s \models \varphi$.

1. $M, s \models p$, where p is a primitive proposition, if p is true under the truth assignment $\pi(s)$.
2. $M, s \models \sim\varphi$ if $M, s \not\models \varphi$.
3. $M, s \models \varphi_1 \wedge \varphi_2$ if $M, s \models \varphi_1$ and $M, s \models \varphi_2$.
4. $M, s \models \Box_i \varphi$ if $M, t \models \varphi$ for all t such that $(s,t) \in \mathcal{R}(i)$.

The following theorem provides an exact correspondence between modal structures and states in Kripke structures.

Theorem 2.4. *To every Kripke structure M and state s in M, there corresponds a modal structure $\mathfrak{f}_{M,s}$ such that $M,s \models \varphi$ iff $\mathfrak{f}_{M,s} \models \varphi$, for every formula φ. Conversely, there is a Kripke structure M such that for every modal structure $\mathfrak{f}$ there is a state $s_{\mathfrak{f}}$ in M such that $\mathfrak{f} \models \varphi$ iff $M, s_{\mathfrak{f}} \models \varphi$, for every formula φ.*

Proof. Suppose $M = (S, \pi, \mathscr{R})$ is a Kripke structure. For every state s in M we construct a modal structure $\mathfrak{f}_{M,s} = < s_0, s_1, ... >$. s_0 is just the truth assignment $\pi(s)$. Suppose we have constructed $s_0, s_1, ..., s_k$ for each state s in M. Then $s_{k+1}(i) = \{<t_0,...,t_k>: (s,t) \in \mathscr{R}(i)\}$. We leave it to the reader to check that $M,s \models \varphi$ iff $\mathfrak{f}_{M,s} \models \varphi$.

For the converse, let $M = (S, \pi, \mathscr{R})$, where S consists of *all* the modal structures, $\pi(\mathfrak{f}) = f_0$, and $(\mathfrak{f}, \mathfrak{g}) \in \mathscr{R}(i)$ iff $<g_0,...g_k> \in f_{k+1}(i)$ for every $k \geq 0$. As before, $M, \mathfrak{f} \models \varphi$ iff $\mathfrak{f} \models \varphi$. ■

Theorem 2.4 shows that modal structures and Kripke structures have the same theory (an analogous result for Kripke structures and modal algebras was shown in [Bi,JT,Lem1]), but its implication are deeper. It shows that modal structures model particular possible worlds, while Kripke structures model collections of possible worlds.

We note that Kripke structures are "flabby" in the sense that two nonisomorphic Kripke structures can be semantically equivalent. However, this is not the case for modal structures.

Decidability. We say that φ is *satisfiable* if it is satisfied in some modal structure, and *valid* if it is satisfied in every modal structure. The *validity problem* asks which formulas are valid.

Lemma 2.5. *A formula φ of depth k is satisfiable (respectively, valid) iff some (respectively, every) $(k+1)$-ary world satisfies φ.*

Proof. Assume that $\text{depth}(\varphi) = k$. We now show that φ is valid iff φ holds in every $(k+1)$-ary world $<f_0,...f_k>$. The "satisfiability" part of the lemma then follows easily from the fact that φ is satisfiable iff $\sim\varphi$ is not valid.

If φ holds in every every $(k+1)$-ary world, then φ is valid, since by definition $<f_0,f_1,...> \models \varphi$ iff $<f_0,...f_k> \models \varphi$. Conversely, if w is a $(k+1)$-ary world and $w \not\models \varphi$, then by Theorem 2.2, there is a modal structure $\mathfrak{f}$ with w as a prefix. By definition, $\mathfrak{f} \not\models \varphi$, and so φ is not valid. ■

We now prove decidability.

Theorem 2.6. *The validity problem for modal structures is decidable.*

Proof. Let φ be a formula of depth k whose validity is to be decided. By Lemma 2.5, φ is valid iff φ holds in every $(k+1)$-ary world. Since by Lemma 2.1 there are only a finite number of $(k+1)$-ary worlds to check, this gives us a decision procedure. ■

The extreme simplicity of the above proof of decidability of the validity problem is one of the nice features of modal structures. This simplicity is not present in Kripke's approach, where the standard technique for proving decidability is to show that the logic has the *finite model property.*

Compactness. Just as modal structures give an extremely simple, elegant proof of decidability of the validity problem, they do the same for compactness. Our proof of compactness makes use of the following lemma, which is an easy consequence of König's Infinity Lemma.

Lemma 2.7. *Let T be a rooted tree with finite fanout at each node. If S is a set of nodes of T such that every infinite path in T (beginning at the root) contains a member of S, then there is a finite subset of S with the same property.*

We are now ready to prove the Compactness Theorem.

Theorem 2.8. *Let Σ be a set of formulas. If every finite subset of Σ is satisfiable, then Σ is satisfiable.*

Proof. Let T be a tree, where the kth level of the tree contains all k-ary worlds, and where the parent of the $(k+1)$-ary world $<f_0,...f_k>$ is its k-ary prefix $<f_0,...f_{k-1}>$. Let S be the set of counterexamples to Σ, i.e., $S = \{w: w \not\models \sigma \text{ for some } \sigma \in \Sigma\}$. Let p be an infinite path in T. In the obvious way, p corresponds to a modal structure $\mathfrak{f} = <f_0, f_1, ...>$. Assume that Σ is not satisfiable. Then $\mathfrak{f} \not\models \sigma$ for some $\sigma \in \Sigma$. If $\text{depth}(\varphi) = k$, then by definition the prefix $w = <f_0, ..., f_k>$ does not satisfy σ. Hence, the path p contains $w \in S$. The assumptions of Lemma 2.7 are satisfied (the tree has finite fanout at each node by Lemma 2.1). Hence, there is a finite subset S' of S such that every path in the tree contains a member of S'. This gives us a finite subset of Σ that is not satisfiable. ■

Note that as in the case of decidability of the validity problem, the proof of compactness depends critically on being able to deal with (finite-depth) worlds, instead of (infinite-depth) modal structures. We remark that although the proof used our assumption that there are only a finite number of propositional variables, it is possible to modify the proof to allow an infinite number of propositional variables. (Also, it is easy to see that both the decision problem for validity and completeness are unaffected by there being an infinite number of propositional variables.)

Completeness. We now present a set of axioms and inference rules which we show give a complete axiomatization for the semantics of modal structures.

In later sections, we discuss how to modify the axioms to give a complete axiomatization for "specialized" modal structures which model belief and/or knowledge. There are two axioms (A1) and (A2), and two inference rules (R1) and (R2).

(A1) All substitution instances of propositional tautologies

(A2) $\Box_i(\varphi_1 \Rightarrow \varphi_2) \Rightarrow (\Box_i\varphi_1 \Rightarrow \Box_i\varphi_2)$

(R1) From φ_1 and $\varphi_1 \Rightarrow \varphi_2$ infer φ_2 (*modus ponens*)

(R2) From φ infer $\Box_i\varphi$ (*generalization*)

If there is only one modality, then this system is known as the modal logic K, which is complete for Kripke structures [Ch]. We now prove completeness directly for modal structures.

Before giving the proof, we must introduce some more concepts. Define a new modality $\Diamond_i$ by letting $\Diamond_i\varphi$ be $\sim\Box_i\sim\varphi$. If we interpret $\Box_i p$ as meaning "after program i terminates, p must be true", then $\Diamond_i p$ means "it is possible for program i to terminate with p true". For each k-ary world $w = <f_0, ..., f_{k-1}>$, we now define a formula σ_w of depth $k-1$ which characterizes w. Assume that the propositional variables are $p_1, ..., p_r$. If $w = <f_0>$, then let σ_w be the propositional formula $p_1' \wedge \cdots \wedge p_r'$, where p_j' is p_j if the truth assignment f_0 makes p_j true, and $\sim p_j$ otherwise, for $1 \leq j \leq r$. Assume inductively that σ_w has been defined for each k-ary world w. Recall that W_k is the set of all k-ary worlds. If $\mathscr{S}$ is a set of k-ary worlds, then define $\Box_i!\mathscr{S}$ to be the formula

$$\bigwedge_{w \in \mathscr{S}} \Diamond_i\sigma_w \quad \wedge \quad \bigwedge_{w \in W_k - \mathscr{S}} \Box_i{\sim}\sigma_w$$

In terms of the modal logic of atomic programs, $\Box_i!\mathscr{S}$ means "after program i terminates, the possible k-ary worlds are precisely those in $\mathscr{S}$". If w is the $(k+1)$-ary world $w = <f_0, ..., f_k>$, define σ_w to be the formula $\sigma_{<f_0>} \wedge \bigwedge_i \Box_i!f_k(i)$. (Such formulas are essentially the *normal forms* in [Fi], though we came up with them while still ignorant of [Fi].) Our interest in these formulas stems from the fact that σ_w uniquely describes w in the following sense.

Lemma 2.9. *Let v and w be k-ary worlds. Then $v \models \sigma_w$ if and only if v=w.*

We now prove completeness.

Theorem 2.10. *The system (A1, A2, R1, R2) is a sound and complete axiomatization for modal structures.*

Sketch of Proof. As usual, soundness is straightforward. To show completeness, we show by induction on the depth of formulas γ that if γ is valid, then $\vdash\gamma$ (i.e., γ is *provable*). The base case (when γ is propositional) follows immediately from Axiom (A1). The inductive step (when depth$(\gamma) = k$) is shown by demonstrating the following two claims.

Claim 1. $\vdash\bigvee\{\sigma_w$: w is a $(k+1)$-ary world$\}$.

Claim 2. Let $w = <f_0, ..., f_k>$ be a $(k+1)$-ary world and γ a formula of depth k. Then $w \models \gamma$ iff $\vdash(\sigma_w \Rightarrow \gamma)$.

We now show that completeness follows immediately from these two claims. For, assume that γ is a valid formula of depth k. By Claim 2, we know that $\vdash(\sigma_w \Rightarrow \gamma)$ for every $(k+1)$-ary world w. By Claim 1, we know that $\vdash\bigvee\{\sigma_w$: w is a $(k+1)$-ary world$\}$. By propositional reasoning (using Axiom (A1)), it then follows that $\vdash\gamma$, as desired.

We now need only prove Claims 1 and 2. Claim 2 is proven by a simple induction on the formulas γ. To prove Claim 1, we first prove the following.

Claim 3. $\vdash\bigvee\{\Box_i!\mathscr{S}$: $\mathscr{S}$ is a set of k-ary worlds$\}$.

To prove Claim 3, we replace $\Diamond_i\sigma_w$ by a new primitive proposition q_w. We then see that $\bigvee\{\Box_i!\mathscr{S}$: $\mathscr{S}$ is a set of k-ary worlds$\}$ translates into

$$\bigvee_{\mathscr{S}}\left(\bigwedge_{w \in \mathscr{S}} q_w \quad \wedge \quad \bigwedge_{w \notin \mathscr{S}} {\sim}q_w\right),$$

which is a propositional tautology. Hence, Claim 3 follows by Axiom (A1).

We close by proving Claim 1. By Axiom (A1),

(1) $\vdash\bigvee\{\sigma_{<f_0>}$: f_0 is a truth assignment$\}$.

By (1) and by Claim 3 (once for each modality $\Box_i$), we obtain $\vdash\varphi$, where φ is the formula

$$(\bigvee\{\sigma_{<f_0>}: f_0 \text{ is a truth assignment}\}) \wedge$$

$$\bigwedge_i \bigvee\{\Box_i!\mathscr{S}: \mathscr{S} \text{ is a set of } k\text{-ary worlds}\}.$$

It is straightforward to verify that if we put φ in disjunctive normal form, we obtain $\bigvee\{\sigma_w$: w is a $(k+1)$-ary world$\}$. ■

Our proof of completeness shows that there are quite uniform proofs of valid formulas in our formal system. Thus, if γ is valid and if depth$(\gamma) = k$, then the formal proof begins by showing that $\bigvee\{\sigma_w$: w is a $(k+1)$-ary world$\}$. Then, for each $(k+1)$-ary world w, the formal proof shows that $\sigma_w \Rightarrow \gamma$. The formal proof then invokes Axiom (A1) to prove γ. Again, as in the cases of decidability and compactness, the ability to consider (finite-depth) worlds is critical in our proof of completeness.

Complexity. By Theorem 2.6 the validity problem for modal structures is decidable. Since the number of worlds grows exponentially at each level, the complexity of the procedure described in the proof of that theorem is nonelementary. We prove here that the validity problem for modal structures is

PSPACE-complete. The proof uses the fact that by Theorem 2.4, a formula is satisfied in a modal structure if and only if it is satisfied in a Kripke structure.

Kripke has observed that Kripke structures can be unraveled into trees [Kr2]. This suggests the following procedure to determine satisfiability: for a given formula φ, construct a tree automaton A_φ [Rab,TW] such that A_φ accepts precisely the tree models of φ. Thus φ is satisfiable if and only if A_φ accepts some tree, and the satisfiability problem is reduced to the emptiness problem. This technique was used in [ES,Str,VW] to prove upper bounds for modal logics of programs. Unfortunately, the size of A_φ is exponential in the size of φ, since its state set consists of all sets of subformulas of φ. Since the emptiness problem for tree automata requires polynomial time (it is PTIME-hard), it seems that this technique cannot give us subexponential upper bounds.

There is a fundamental difference, however, between the logic studied here and the logics in [ES,Str,VW]. For those logics, the tree models can be infinite. Here the situation is different. By Lemma 2.5, if a formula φ of depth k is satisfiable then it is satisfiable in a $(k+1)$-ary world. This was indeed the basis for the decidability result of Theorem 2.6. Translated into the framework of Kripke structures, this means that if φ is satisfiable then it is satisfiable in a tree model of depth k. Thus, even though A_φ can be exponentially big, we are interested only in very shallow trees, with depth logarithmic in the size of A_φ. It turns out that to decide whether a tree automaton accepts a shallow tree is easier than to decide whether the automaton accepts some tree: the former problem is in ALOGTIME (alternating logarithmic time), while the latter is PTIME-hard. This enables us to improve the upper bound from EXPTIME to PSPACE.

Theorem 2.11. *The validity problem for modal structures is PSPACE-complete.*

Ladner proved, by analyzing Kripke's tableau-based decision procedure, a PSPACE upper bound for the validity problem for Kripke structures with a single modality [La]. Thus Theorem 2.11 extends his result to the case of multiple modalities. Furthermore, we believe that our technique is not only more elegant than tableau-based decision procedures, but it also has wider applicability. For example, using this technique we can extend all the other PSPACE upper bounds in [La] from the single modality case to multiple modalities (note that the transition from a single modality to multiple modalities may in general increase the complexity of the validity problem [HM2]). Moreover, we can also show that the validity problem for the propositional dynamic logic (PDL) of straight-line programs is PSPACE-complete (the validity problem for PDL of regular programs is known to be EXPTIME-complete [FL,Pra2]).

3. Modelling belief

Belief Structures. Modal structures can be used for modelling many modal logics. As an illustration, we show in this section how to specialize a modal structure so that it models *belief*, and demonstrate the effect this has on our previous theorems and proofs. The nature of belief and its properties has been a matter of great dispute among philosophers (see [Len]). In this section, we concentrate on one natural notion of belief, and note that it is also possible to use modal structures to model various other notions of belief.

We assume that $\mathscr{I}$ is a finite set of "players". For pedagogical reasons, we write the modality $\Box_i$ as B_i, where the formula $B_i\varphi$ means "player i believes φ". We consider the idealized situation where the players are perfect reasoners with perfect introspection, who have consistent world views that may or may not be completely correct. Thus, it is possible for a player to believe something that is not true. This distinguishes belief from knowledge, where whatever a player *knows* is necessarily true. (One can argue that in the real world, there is rarely, if ever, knowledge, but only belief.) By "perfect introspection", we mean that if a player believes something, then he believes that he believes it, and if he does not believe something, then he believes that he does not believe it. Further, each player believes that each of the other players is also a perfect reasoner.

Belief worlds and *belief structures* are defined just as we defined *worlds* and *modal structures* before, except that there are two additional semantical restrictions on belief worlds $<f_0, \ldots, f_k>$:

1. $f_k(i)$ is nonempty, for each $k \geq 1$ and each player i. This restriction says that player i believes that *some* k-ary belief world is possible, for each k. This corresponds to our intuitive notion above which said that each player has a consistent world view.
2. If $<g_0, \ldots g_{k-1}> \in f_k(i)$, and $k > 1$, then $g_{k-1}(i) = f_{k-1}(i)$. This corresponds to our intuitive notion above, which said that each player is introspective. Thus, player i will not consider a belief world w possible unless his beliefs as encoded in w are his actual beliefs. In other words, let S be the set of $(k-2)$-ary belief worlds that player i believes possible, that is, $S = f_{k-1}(i)$. Then player i will not consider a

$(k-1)$-ary belief world w possible unless w "says" that S is the set of possible $(k-2)$-ary belief worlds for player i. That is, if $<g_0, ..., g_{k-1}> \in f_k(i)$, then $g_{k-1}(i) = S$.

The proofs of decidability and compactness of the previous section go through exactly as before. The only non-trivial part is to replace Theorem 2.2 by the following theorem.

Theorem 3.1. *Each belief world is the prefix of a belief structure.*

Proof. Let $w = <f_0, ..., f_{k-1}>$ be a belief world. If $k = 1$, then define $f_k(i) = \{<g_0> : g_0$ is a truth assignment$\}$ for each player i. If $k > 1$, then define $f_k(i) = \{<g_0, ..., g_{k-1}> : g_{k-1}(i) = f_{k-1}(i)$ and $<g_0, ... g_{k-2}> \in g_{k-1}(i)\}$ for each player i. Intuitively, $f_k(i)$ contains every $(k-1)$-ary belief world consistent with player i's beliefs. Similarly, define f_{k+1}, f_{k+2}, etc. We can then verify that $<f_0, ... f_{k-1}, f_k, ... >$ is a belief structure, with w as a prefix. ∎

In addition to the axioms and inference rules of the previous section (where we replace $\Box_i$ by B_i), we need three new axioms to reflect the new semantic constraints on a belief structure.

(B1) $\sim B_i$(false) ("Player i does not believe a contradiction").

(B2) $B_i\varphi \Rightarrow B_iB_i\varphi$ ("Introspection of positive belief").

(B3) $\sim B_i\varphi \Rightarrow B_i{\sim}B_i\varphi$ ("Introspection of negative belief").

Theorem 3.2. *The system (A1, A2, B1, B2, B3, R1, R2) is a sound and complete axiomatization for belief structures.*

Proof. The proof of completeness is just as before, except that it is necessary to slightly extend the proof of Claim 1, as follows. We must show that if w is a world that is not a belief world, then $\vdash \sim\sigma_w$. This last step is precisely where we make use of the new axioms (B1), (B2), and (B3). ∎

Levesque axiomatized belief for one player [Lev]. Our axiomatization shows that one of Levesque's axioms is redundant.

Modal structures are used in [FHV] to model knowledge. In the full paper, we show how to model knowledge and belief simultaneously by modal structures. As in the case of belief, our previous theorems and proofs go through with natural modifications.

Relationship to Kripke Structures. To model belief by Kripke structures, we have to restrict the class of binary relations that are assigned to the modalities. A relation R is *euclidean* if $(y,z) \in R$ whenever $(x,y) \in R$ and $(x,z) \in R$. R is *serial* (with respect to the set S) if for each $s \in S$ there is t such that $(s,t) \in R$. A Kripke structure $(S, \pi, \mathscr{R})$ is a *Kripke belief structure* if $\mathscr{R}(i)$ is euclidean and serial (with respect to S) for each player i [Sta].

The relationship between belief structures and Kripke belief structures is analogous to the relationship between modal structures and Kripke structures.

Theorem 3.3. *To every Kripke belief structure M and state s in M, there corresponds a belief structure $\mathfrak{f}_{M,s}$ such that $M, s \models \varphi$ iff $\mathfrak{f}_{M,s} \models \varphi$, for every formula φ. Conversely, there is a Kripke belief structure M such that for every belief structure $\mathfrak{f}$ there is a state $s_{\mathfrak{f}}$ in M such that $\mathfrak{f} \models \varphi$ iff $M, s_{\mathfrak{f}} \models \varphi$, for every formula φ.*

The reader should note the intuitiveness in the definition of belief structures compared to the nonintuitiveness in the definition of Kripke belief structures. Because of this nonintuitiveness, it is not clear how to model particular states of belief by Kripke belief structures (cf. [FHV]). In the full paper we shall demonstrate how to do such modelling with belief structures. We also note that modelling knowledge and belief by modal structures brings up new concepts, such as "finite amount of information" [FHV]. All of these issues are further evidence for the advantage of modal structures over Kripke structures in conceptual modelling.

Complexity. In trying to apply our automata-theoretic technique to the decision problem for belief structures, we encounter a difficulty: because of the restriction on the binary relations in Kripke belief structures, the tree models for belief formulas may not be shallow. To get around that difficulty we first reduce the decision problem for belief structures to the decision problem for a certain fragment of propositional dynamic logic. This class does have shallow tree models, so our technique is applicable. It turns out that for belief the difference between a single modality and multiple modalities is crucial.

Theorem 3.4.

1. *The decision problem for belief structures with a single player is NP-complete.*
2. *The decision problem for belief structures with at least two players is PSPACE-complete.*

Similar results were shown for knowledge in [HM2] by different techniques. In the full paper we show that these bounds also hold for the combined knowledge-belief structures.

4. Modelling Knowledge and Joint Knowledge

In this section we consider a more idealized situation in which *knowledge* is possible. (As mentioned before, the difference between belief and knowledge is that whatever is *known* must be true.) We now write the modality $\Box_i$ as K_i, where the formula $K_i\varphi$

means "player i knows φ". Again we consider the players to be perfect reasoners with perfect introspection.

We are mostly interested here in *joint knowledge*, the knowledge shared by a set of players. When we say that φ is joint knowledge of players 1 and 2, we mean more than just that both 1 and 2 know φ; we require also that 1 knows that 2 knows φ, 2 knows that 1 knows φ, 1 knows that 2 knows that 1 knows φ, and so on. For example, if 1 and 2 are both present when a certain event happens and see each other there, then the event become joint knowledge. The notions of joint knowledge has applications in game theory (cf. [Ha]), econometrics (cf. [Rad]), artificial intelligence (cf. [MSHI]), and distributed computing (cf. [HM1]).

To be able to reason about joint knowledge, Fagin et al. [FHV] extend the logic by modalities $C_{\mathscr{P}}$ for every set $\mathscr{P}$ of players, where intuitively $C_{\mathscr{P}}\varphi$ means "φ is joint knowledge of the players in $\mathscr{P}$". We do not exclude the case where $\mathscr{P}$ is the empty set of players; in this case, $C_{\emptyset}\varphi$ turns out to be equivalent to φ. Joint knowledge between the set of *all* players is known as *common knowledge* [FHV, HM1, HM2, Leh]. To assign semantics to the new modalities, we first define $E_{\mathscr{P}}\varphi$ as a shorthand for $\bigwedge\{K_i\varphi: i \in \mathscr{P}\}$ i.e., "everybody in $\mathscr{P}$ knows φ". We then take $C_{\mathscr{P}}\varphi$ to stand for the infinite conjunction $E_{\mathscr{P}}\varphi \wedge E_{\mathscr{P}}E_{\mathscr{P}}\varphi \wedge \ldots$.

Taken this way, the depth of formulas in the extended logic is not finite anymore. To be able to define satisfaction for such formulas, Fagin et al. [FHV] define knowledge structures of transfinite "length". (In their terminology, modal structures as defined in Section 1 are ω-worlds, and to extend them they define $f_\lambda(i)$ to be a set of λ-worlds, for each ordinal λ.)

Since now we have formulas and worlds of infinite depth, it may seem that our approach would not be applicable to the logic of joint knowledge. (Indeed, it is not hard to verify that compactness fails here.) There is, however, a way around that difficulty. Rather than take $C_{\mathscr{P}}\varphi$ as a shorthand for the infinite conjunction $E_{\mathscr{P}}\varphi \wedge E_{\mathscr{P}}E_{\mathscr{P}}\varphi \wedge \ldots$, we take $C_{\mathscr{P}}$ to be a modality it its own right. This enables us to view the formulas as having only finite depth, Thus, rather than use kth order assignments, which are functions $f_k: \mathscr{P} \rightarrow 2^{W_k}$, we use *augmented* kth order assignments, which are functions $f_k: 2^{\mathscr{P}} \rightarrow 2^{W_k}$. That is, we assign a set of possible worlds to every set of players. *Augmented worlds* and *augmented structures* are defined analogous to *worlds* and *structures* with augmented assignments rather than the standard assignments.

To make these worlds and structures models for joint knowledge, we have to impose several semantic restrictions. Before listing the restrictions we need some definitions. Let $w_1 = \langle f_0, \ldots, f_{k-1} \rangle$ and $w_2 = \langle g_0, \ldots, g_{k-1} \rangle$ be two k-ary augmented worlds, and let $\mathscr{P}$ be a set of players. We say that w_1 and w_2 are *equivalent with respect to* $\mathscr{P}$, denoted $w_1 \equiv_{\mathscr{P}} w_2$, if either $k=1$, or $k>1$ and for every set $\mathscr{Q}$ of players such that $\mathscr{P} \subseteq \mathscr{Q}$ we have that $f_{k-1}(\mathscr{Q}) = g_{k-1}(\mathscr{Q})$. That is, w_1 and w_2 are equivalent with respect to $\mathscr{P}$ if, as far as the players in $\mathscr{P}$ are concerned, these worlds are identical. Let $w_1 = \langle f_0, \ldots, f_{r-1} \rangle$ and $w_2 = \langle g_0, \ldots, g_{k-1} \rangle$ be augmented worlds, and let $\mathscr{P}$ be a set of players. We say that w_2 *$\mathscr{P}$-appears in* w_1 if either $k \le r$ and $w_2 = \langle f_0, \ldots, f_{k-1} \rangle$ or there are a player $i \in \mathscr{P}$ and an augmented world $w_3 \in f_{r-1}(\{i\})$ such that w_2 $\mathscr{P}$-appears in w_3. Intuitively, w_2 $\mathscr{P}$-appears in w_1 if in w_1 some player in $\mathscr{P}$ thinks it possible that some player in $\mathscr{P}$ thinks it possible ... that w_2 is possibly the actual world.

A $(k+1)$-ary augmented world $\langle f_0, \ldots, f_k \rangle$ must satisfy the following restrictions in order to be a *joint knowledge world*:

1. $f_k(\emptyset)$ is nonempty. In fact, along with condition (3) below, this condition guarantees that $f_k(\emptyset)$ is the singleton set $\{\langle f_0, \ldots, f_{k-1} \rangle\}$ which contains only the "real" world.
2. If $\mathscr{P}$ and $\mathscr{Q}$ are sets of players such that $\mathscr{P} \subseteq \mathscr{Q}$, then $f_k(\mathscr{P}) \subseteq f_k(\mathscr{Q})$ for $k \ge 1$. This captures the intuition that the less players there are, the easier it is for them to share knowledge.
3. If $\langle g_0, \ldots, g_{k-1} \rangle \in f_k(\mathscr{P})$, and $k>1$, then $\langle g_0, \ldots, g_{k-1} \rangle \equiv_{\mathscr{P}} \langle f_0, \ldots, f_{k-1} \rangle$. That is, the players know exactly what they know. This restriction is analogous to restriction (2) for belief worlds.

Lemma 4.1. *If $\langle f_0, \ldots, f_k \rangle$ is a joint knowledge world and $k \ge 1$, then $\langle f_0, \ldots, f_{k-1} \rangle \in f_k(\mathscr{P})$ for each set $\mathscr{P}$ of players.*

Proof. As we noted, conditions (1) and (3) guarantee that $f_k(\emptyset) = \{\langle f_0, \ldots, f_{k-1} \rangle\}$. Condition (2) guarantees that $f_k(\emptyset) \subseteq f_k(\mathscr{P})$ for $k \ge 1$. The lemma follows immediately. ■

Lemma 4.1 says that the "real" world is always one of the possibilities, for every set $\mathscr{P}$ of players. This ensures that what is known must be true.

Joint knowledge structures are defined from joint knowledge worlds in the now standard way, but in addition they have to satisfy the following restriction:

4. If $\langle f_0 f_1 f_2, \ldots \rangle$ is a joint knowledge structure and $\langle g_0, \ldots, g_{k-1} \rangle \in f_k(\mathscr{P})$ for some set $\mathscr{P}$ of players, then there exists an r such that $\langle g_0, \ldots, g_{k-1} \rangle$ $\mathscr{P}$-appears in $\langle f_0, \ldots, f_{r-1} \rangle$.

This restriction ensures that the semantics of $C_{\mathscr{P}}\varphi$ is indeed that everybody in $\mathscr{P}$ knows that everybody in $\mathscr{P}$ knows ... that φ.

In the full paper we show that the notion of joint knowledge captured by joint knowledge structures is equivalent to that captured in the "long" knowledge structures in [FHV] and in the Kripke structures in [HM2].

Restriction (4) is different from the other restrictions, because it is a restriction on structures and not on worlds. As a result of this restriction, the proof of Theorem 2.8 breaks down, and the logic indeed is not compact. Furthermore, because of this restriction the analogue to Theorem 2.2 and Theorem 3.1 also fails. It may seem that our methodology fails completely for joint knowledge. Fortunately, we can prove a somewhat weaker version of Theorem 2.2 and Theorem 3.1. Let $\mathscr{C}$ be a set of k-ary joint knowledge worlds, and let $\mathscr{P}$ be a set of players. The *$\mathscr{P}$-graph* of $\mathscr{C}$ has the worlds in $\mathscr{C}$ as nodes, and it has an edge between two worlds w_1 and w_2 if $w_1 \equiv_{\{i\}} w_2$ for some player $i \in \mathscr{P}$. We say that $\mathscr{C}$ is *$\mathscr{P}$-connected* if the $\mathscr{P}$-graph of $\mathscr{C}$ is connected. Let $w = \langle f_0, \ldots, f_k \rangle$ be a joint knowledge world. We say that w is *connected* if $f_k(\mathscr{P})$ is $\mathscr{P}$-connected for every set $\mathscr{P}$ of players.

Theorem 4.2. *A joint knowledge world is the prefix of a joint knowledge structure if and only if it is connected.*

By Theorem 4.2, it suffices to consider only connected joint knowledge worlds. Decidability of the validity problem for joint knowledge structures easily follows. Our technique to prove completeness extends also (using Theorem 4.2) to joint knowledge structures to yield a complete axiomatization. It is convenient to have in our logic only modalities of the form $C_{\mathscr{P}}$ (where K_i is considered simply an abbreviation for $C_{\{i\}}$). In addition to the axioms and inference rules of Section 2 (where we replace $\Box_i$ by $C_{\mathscr{P}}$), we need the following new axioms:

(J1) $C_{\emptyset}\varphi \equiv \varphi$.
(J2) $C_{\mathscr{Q}}\varphi \Rightarrow C_{\mathscr{P}}\varphi$ if $\mathscr{P} \subseteq \mathscr{Q}$.
(J3) $C_{\mathscr{P}}\varphi \Rightarrow C_{\mathscr{P}}C_{\mathscr{P}}\varphi$ ("positive introspection").
(J4) $\sim C_{\mathscr{P}}\varphi \Rightarrow C_{\mathscr{P}}\sim C_{\mathscr{P}}\varphi$ ("negative introspection").
(J5) $C_{\mathscr{P}}(\varphi \Rightarrow E_{\mathscr{P}}\varphi) \Rightarrow (\varphi \Rightarrow C_{\mathscr{P}}\varphi)$ ("induction axiom").

Theorem 4.3. *The system (A1, A2, J1, J2, J3, J4, J5, R1, R2) is a sound and complete axiomatization for joint knowledge structures.*

The above axiomatization for joint knowledge generalizes known axiomatizations for common knowledge [HM2,Leh]. Unlike the previous logics, with joint knowledge, however, we no longer have shallow tree models. By reducing the decision problem to that of propositional dynamic logic, we can prove an exponential time upper bound, which matches Halpern and Moses' exponential time lower bound [HM2].

5. Acknowledgments

We are grateful to Joe Halpern, Yoram Moses, Ed Wimmers, and Pierre Wolper, who read previous drafts of this paper and made helpful comments. We would also like to thank the participants at the CSLI seminar for their suggestions.

Part of the research reported here was done while the second author was visiting the IBM San Jose Research Laboratory. The second author's research at CSLI was supported by a gift from System Development Foundation.

6. References

[Bat] Bates, B.: Leibniz on possible worlds. *Logic, Methodology, and Philosophy of Science III* (B. van Rootselaar and J.F. Staal, eds.), North-Holland, 1968, pp. 507-530.

[Bay] Bayart, A.: La correction de la logique modale du premier et second ordre S5. *Logique et Analyse* 1(1958), pp. 28-44.

[Bi] Birkhoff, G.: Rings of sets. *Duke Math. J.* 3(1937), pp. 443-454.

[Bo] Bochmann, G.V.: Hardware specification with temporal logic: an example. *IEEE Trans. on Computers* C-31(1982), pp. 223-231.

[Bu] Burgess, J.P.: Decidability for branching time. *Studia Logica* 39(1980), pp. 203-218.

[Ca1] Carnap, R.: Modalities and quantification. *J. Symbolic Logic* 11(1946), pp. 33-64.

[Ca2] Carnap, R.: *Meaning and necessity.* The University of Chicago Press, 1947.

[CCF] Castilho, J.M.V., Casanova, M.A., Furtado, A.L.: A temporal framework for database specification. *Proc. 8th Int. Conf. on Very Large Data Bases*, Mexico City, 1982, pp. 280-291.

[CES] Clarke, E.M., Emerson, E.A., Sistla, A.P.: Automatic verification of finite state concurrent systems using temporal logic specifications - a practical approach. *Proc. 9th ACM Symp. on Principles of Programming Languages*, 1982, pp. 117-126.

[Ch] Chellas, B.F.: *Modal logic.* Cambridge Univ. Press, 1980.

[ES] Emerson, E.A., Sistla, A.P.: Deciding branching time logic. *Proc. 16th ACM Symp. on Theory of Computing*, 1984, pp. 14-24.

[FHV] Fagin, R., Halpern, J.Y., Vardi, M.Y.: A model-theoretic analysis of knowledge - preliminary report. *Proc. 25th IEEE Symp on Foundations of Computer Science*, 1984, pp. 268-278.

[Fi] Fine, K.: Normal forms in modal logic. *Notre Dame J. Formal Logic* 16(1975), pp. 229-237.

[FL] Fischer, M., Ladner, R.E.: Propositional dynamic logic of regular programs. *J. Computer and System Sciences* 18(1979), pp. 194-211.

[Fr] van Fraassen, B.: Compactness and Lowenheim-Skolem proofs in modal logic. *Logique et Analyse* 12(1969), pp. 167-178.

[Go] Goldblatt, R.I.: First-order definability in modal logic. *J. Symbolic Logic* 40(1975), pp. 35-40.

[Gu] Guillaume, M.: Rapports entre calculs propositionnels modaux et topologie impliques par certaine extensions de la methode des tableaux semantiques: Systeme de Feys-von Wright, Systeme S4 de Lewis, Systeme S5 de Lewis. *Comptes Rendues des Seances de l'Academie de Sciences (Paris)* 246(1958), pp. 1140-1142, 246(1958), pp. 2207-2210, and 247(1958), pp. 1282-1283.

[HM1] Halpern, J.Y., Moses, Y.: Knowledge and common knowledge in distributed environments. *Proc. 3rd ACM Symp. on Principles of Distributed Computing* (1984), 50-61.

[HM2] Halpern, J.Y., Moses, Y.: *A guide to the modal logic of knowledge*. To appear.

[Ha] Harsany, J.: Games of incomplete information played by Bayesian players, parts I-III. *Management Science* 14(1967-8), pp. 159-182, 320-334, 486-502.

[Hi1] Hintikka, J.: Quantifiers in deontic logic. *Societas Scientiarumi Fennica, Commentationes Humanarum Literarum* 23,4(1957), 1-23.

[Hi2] Hintikka, J.: Modalities and quantification. *Theoria*, 27(1961), 119-128.

[JT] Jonsson, B., Tarski, A.: Boolean algebras with operators I. *American J. of Math.* 23(1951), pp. 891-939.

[Ka] Kanger, S.: *Provability in Logic*. Stockholm Studies in Philosophy I, 1957.

[KP] Kozen, D., Parikh, R.: An elementary proof of the completeness of PDL. *Theoretical Computer Science*, 14(1981), pp. 113-118.

[Kr1] Kripke, S.A.: A completeness theorem in modal logic. *J. Symbolic Logic* 24(1959), pp. 1-14.

[Kr2] Kripke, S.A.: Semantical analysis of modal logic I - normal modal propositional calculi. *Zeit. f. math. Logik und Grundlagen d. Math.* 9(1963), pp. 67-96. (Announced in *J. Symbolic Logic* 24(1959), p. 323.)

[La] Ladner, R.E.: The computational complexity of provability in systems of modal propositional logic. *SIAM J. Comput.* 6(1977), pp. 467-480.

[Leh] Lehmann, D.: Knowledge, common knowledge, and related puzzles. *Proc. 3rd ACM Symp. on Principles of Distributed Computing*, 1984, Vancouver, pp. 62-67.

[Lem1] Lemmon, E.J.: Algebraic semantics for modal logics, parts I-II. *J. Symbolic Logic* 31(1966), pp. 45-65 and pp. 191-218.

[Lem2] Lemmon, E.J.: An introduction to modal logic (the "Lemmon Notes"), in collaboration with D. Scott and edited by K. Segerberg. *American Phil. Quarterly Monograph Series #11*, 1977.

[Len] Lenzen, W.: Recent work in epistemic logic. *Acta. Phil. Fennica* 30(1978), pp. 1-218.

[Lev] Levesque, Hector J.: *A formal treatment of incomplete knowledge bases*, Fairchild Technical Report No. 614, FLAIR Technical Report No. 3, Feb. 1982.

[Li] Lipski, W.: On databases with incomplete information. *J. ACM* 28(1981), pp. 41-70.

[Ma] Makinson, D.: On some completeness theorems in modal logic. *Zeit. f. math. Logik und Grundlagen d. Math.* 12(1966), pp. 379-384.

[MW] Manna Z., Wolper, P.: Synthesis of communicating processes from temporal logic specifications. *ACM Trans. Programming Languages and Systems* 6(1984), pp. 68-93.

[MH] McCarthy, J., Hayes, P.: Some philosophical problems from the standpoint of artificial intelligence. In *Machine Intelligence 4* (D. Michie, ed.), American Elsevier, 1969, pp. 463-502.

[MSHI] McCarthy, J., Sato, M., Hayashi, T., Igarashi, S.: *On the model theory of knowledge*. Stanford AI Lab Memo AIM-312, Stanford University, 1978.

[Mc] McKinsey, J. C. C.: A solution of the decision problem for the Lewis Systems S2 and S4, with an application to topology. *J. Symbolic Logic* 6(1941), pp. 117-134.

[McT] McKinsey, J. C. C., Tarski, A.: Some theorems about the sentential calculi of Lewis and Heyting. *J. Symbolic Logic* 13(1948), pp. 83-96.

[Me] Meredith, C.A.: *Interpretations of different modal logics in the "Property Calculus"*. Mimeographed manuscript, Philosophy Department, Canterbury University College (recorded and expanded by A.N. Prior), 1956.

[Mo1] Montague, R.: Logical necessity, physical necessity, ethics, and quantifiers. *Inquiry* 4(1960), pp. 259-269. (Originally delivered before the Ann. Spring Conf. in Phil. at the Univ. of California, Los Angeles, 1955).

[Mo2] Montague, R.: Pragmatics. In *Contemporary Philosophy* (R. Kalibansky, ed.), La Nuova Italia Editrice, Florence, 1968, pp. 102-121.

[Pn] Pnueli, A.: The temporal logic of programs. *Proc. 18th IEEE Symp. on Foundation of Computer Science*, 1977, pp. 46-57.

[Pra1] Pratt, V.R.: Semantical considerations on Floyd-Hoare logic. *Proc. 17th IEEE Symp. on Foundations of Computer Science*, Houston, 1976, pp. 109-121.

[Pra2] Pratt, V.R.: A near optimal method for reasoning about action. *J. Computer and System Sciences* 20(1980), pp. 231-254.

[Pri] Prior, A.N.: Possible worlds (idea attributed to P.T. Geach). *Phil. Quarterly* 12(1962), pp. 36-43.

[Rab] Rabin, M.O.: Decidability of second-order theories and automata on infinite trees. *Trans. AMS* 141(1969), pp. 1-35.

[Rad] Radner, R.: Competitive equilibrium under uncertainty. *Econometrica* 36(1968), pp. 21-58.

[Re] Rescher, N.: On alternatives in epistemic logic. In *Studies in Modality, American Phil. Quarterly Monograph Series #8*(1974), pp. 99-114.

[RS] Reif, J.H., Sistla, A.P.: A multiprocessor network logic with temporal and spatial modalities. *Proc. 10th Int. Colloq. on Automata, Languages and Programming*, Barcelona, 1983, Springer-Verlag, Lecture Notes in Computer Science - Vol. 154.

[Sc] Scott, D.: Advice on modal logic. In *Philosophical problems in logic* (K. Lambert, ed.), Reidel, 1970, pp. 143-173.

[SM] Schwartz, R.L., Melliar-Smith, P.M.: Temporal specification of distributed systems. *Proc. 2nd Int. Conf. on Distributed Systems*, Paris, 1981, pp. 1-9.

[Sta] Stalnaker, R.: *A note on non-monotonic logic*. Dept. of Philosophy, Cornell University, unpublished manuscript, 1980.

[Str] Streett, R.S.: Propositional dynamic logic of looping and converse is elementarily decidable. *Information and Control* 54(1982), pp. 121-141.

[Ts] Tsao-Chen, T.: Algebraic postulates and a geometric interpretation for the Lewis calculus of strict implication. *Bulletin American Math. Soc. (Series 2)* 44(1938), pp. 737-744.

[TW] Thatcher, J.W., Wright, J.B.: Generalized finite automata theory with an application to a decision problem of second-order logic. *Math. System Theory* 2(1968), pp. 57-81.

[VW] Vardi, M.Y., Wolper, P.: Automata theoretic-techniques for modal logic of programs. *Proc. 16th ACM Symp. on Theory of Computing*, 1984, pp. 446-456.