SIAM J. DISCRETE MATH.
Vol. 30, No. 4, pp. 1978–1996

AN ALGORITHMIC VIEW OF VOTING*

RONALD FAGIN†, RAVI KUMAR‡, MOHAMMAD MAHDIAN‡, D. SIVAKUMAR‡, AND ERIK VEE‡

Abstract. We offer a novel classification of voting methods popular in social choice theory. Our classification is based on the more general problem of *rank aggregation* in which, beyond electing a winner, we also seek to compute an aggregate ranking of all the candidates; moreover, our classification is offered from a computational perspective—based on whether or not the voting method generalizes to an aggregation algorithm guaranteed to produce solutions that are near optimal in minimizing the distance of the aggregate ranking to the voters' rankings with respect to one of three well-known distance measures: the Kendall tau, the Spearman footrule, and the Spearman rho measures. We show that methods based on the average rank of the candidates (Borda counting), on the median rank of the candidates, and on the number of pairwise-majority wins (Copeland) all satisfy the near-optimality criterion with respect to each of these distance measures. On the other hand, we show that natural extensions of each of plurality voting, single transferable voting, and Simpson–Kramer minmax voting do not satisfy the near-optimality criterion with respect to these distance measures.

Key words. rank aggregation, social choice, approximation algorithms

AMS subject classifications. 91-08, 91B12, 91B14, 68W25

DOI. 10.1137/15M1046915

1. Introduction.

One of the crowning achievements of microeconomic theory during the past century is the formal treatment of issues pertaining to social welfare. A prime specific example is the theory of social choice, which addresses—in an axiomatic mathematical framework—the question of what constitutes a good method for ranking the candidates in an election where voters present their individual rankings of a set of candidates.

The earliest forms of elections, including, for example, the elections of leaders of various tribes, were based largely on the *plurality* method, where the candidate with the most (first-place) votes was declared the winner. According to the plurality method (in the absence of ties for first place), it does not matter what the voters' complete preference orders are. During the 1780s, two Frenchmen—Jean-Charles de Borda and Nicolas de Condorcet—challenged the ancient wisdom of plurality elections, and argued forcefully why it was important to consider the entire preference orders even if the goal is only to choose the winner of the election.

Borda [4] proposed an extension of plurality where candidates are ordered by their average ranks. Condorcet [5] pointed out a systematic weakness of the plurality method as well as that of any scoring method such as Borda's: they could elect a candidate B as the winner even though there is another candidate A that a majority prefers to B. He proposed the idea, now known as the *Condorcet criterion*, that if there is a candidate A such that for each other candidate B, a majority of the

*Received by the editors November 4, 2015; accepted for publication (in revised form) August 16, 2016; published electronically October 19, 2016. Part of this work was done while all authors were at IBM Almaden.
http://www.siam.org/journals/sidma/30-4/M104691.html
†IBM Almaden, San Jose, CA 95120 (fagin@us.ibm.com).
‡Google, Inc., Mountain View, CA 94043 (ravi.k53@gmail.com, mahdian@gmail.com, dsivakumar@gmail.com, erikvee@gmail.com).

voters prefer A to B, then candidate A shall be the winner. The Condorcet criterion, however, is not a decisive voting method: it is possible that there is no candidate satisfying the hypothesis of the criterion, for example, three candidates A, B, and C such that a majority prefers A to B, a majority prefers B to C, and a majority prefers C to A. The resulting tension was at the heart of much debate in social choice theory during the latter half of the 20th century, when the subject received more mathematical treatment.

In parallel, a number of elegant voting mechanisms such as Simpson–Kramer min-max voting and single transferable voting (STV) have gained popularity for a variety of reasons. These mechanisms are akin to plurality in that they consider only "extreme" data (most first place votes, smallest largest margin of pairwise defeat, eliminate candidate with fewest first place votes and iterate, etc.).

The present work offers a novel perspective from the viewpoint of polynomial-time approximation algorithms. To enable this viewpoint, we study the more general problem of *rank aggregation*, in which the task is not just to elect a winner but to produce a complete aggregate ranking of the given input choices. Specifically, we study natural optimization problems of the following form, as suggested by Kemeny [11]: given a number of input rankings on a set of candidates, compute a ranking whose total distance to the given rankings is minimized. The choice of distance functions we consider include the Kendall tau distance (preferred by Kemeny), the Spearman footrule distance, and the Spearman rho distance (definitions to be presented shortly).

The main results of this paper can be summarized as follows: for the rank aggregation problem, sorting candidates by their average rank (the method of Borda), sorting the candidates by their median ranks, and sorting candidates by the number of pairwise-majority wins (proposed by Copeland), all yield constant-factor polynomial-time approximation algorithms for the minimization problem with respect to all the aforementioned distance measures. On the other hand, natural extensions of each of plurality, Simpson–Kramer min-max, and STV to aggregation problems do not yield approximation algorithms (polynomial time or not) for these minimization problems. As a tool for one of our proofs, we prove a surprising identity about permutations (Theorem 4.8), that is interesting in its own right.

In the rest of this introduction, we discuss the importance of the minimization problem with respect to the Kendall tau distance; clarify the relationships among the distance measures; and point out relevant prior results and how our contributions relate to prior work.

1.1. Kendall tau minimization. While Borda and Condorcet had pioneered the idea that voting schemes (whose goal is to elect a winner) should consider all the information present in the voters' preference orders, in the 20th century Arrow [2] studied the question under the broader umbrella of aggregation (whose goal is to aggregate the voters' rankings into a complete ranking of the candidates). The basic principle underlying Arrow's work is in the spirit of Condorcet's criterion, and requires that the relative positions of two candidates in the aggregation should depend only on their relative positions among the voters' lists. Arrow established the deep result [2] that rules out all reasonable methods for aggregation; this is essentially a very powerful manifestation of the cycle dilemma that arises from Condorcet's criterion.

In the wake of Arrow's result, Kemeny [11] proposed an aggregation method where the objective is to find a ranking that minimizes the total *Kendall tau* distance [12] to the voters' rankings. The Kendall tau distance between two rankings σ and π, denoted $K(\sigma, \pi)$, is defined as the number of "upsets" in π with respect to σ, that is, the number of pairs of candidates whose relative ordering is different between σ and π. Kemeny's proposal turned out to unify the Borda and Condorcet camps to some degree: Young and Levenglick [17] established that Kemeny's proposal is the unique aggregation method that meets the Condorcet criterion and also satisfies two of the three desirable social-choice properties (neutrality[1] and consistency,[2] excluding anonymity[3]) that characterize Borda's method.

From a computational viewpoint, it was shown [3] that computing solutions that minimize the total Kendall tau distance is NP-hard,[4] and hence likely to be computationally intractable; in fact, it remains so even when there are just four voters [8]. The latter paper introduced Kendall-tau optimality (and Condorcet-type criteria) as a useful paradigm in rank aggregation problems that arise from web search and information-retrieval applications. In these situations (quite unlike traditional elections), the number of candidates is large, and the number of voters is relatively modest.

1.2. Objective functions and approximation problems. Taking "total Kendall tau distance" as a reasonable objective function, one may pose two types of problems: finding efficient algorithms that achieve approximation factors as close to 1 as possible, and establishing that well-known aggregation proposals (such as the Borda method, Copeland method, etc.) achieve constant approximation factors (that are independent of the number of candidates or voters). The former class of problems is important from the viewpoint of applications of rank aggregation; the state of the art is an algorithm of [1, 16, 13] that produces aggregations whose total Kendall tau distance to the voters' rankings is at most a factor $(1+\epsilon)$ worse than the total distance achieved by an optimal aggregation for any constant $\epsilon > 0$.

Our goal here is the latter, namely, to establish that the proposals of Borda and Copeland, as well as a natural variant of Borda (using the median instead of the mean), achieve constant approximation factors with respect to optimal solutions in the Kemeny sense. We deal with n candidates and k voters and we are interested in ranking the entire set of n candidates, rather than just electing one winner.

Spearman proposed [15] two distances other than the Kendall tau distance between two rankings. These distance measures make use of the L_1 and L_2 metrics: the *Spearman footrule distance* and the *Spearman rho distance* between two rankings σ and π is the sum, over all the candidates, of the "distance" in the position of the candidate in both the rankings (distance is either the absolute distance or the squared distance, respectively, in the two cases). It was observed in [8] that a ranking that optimizes the Spearman footrule or the Spearman rho distance to the input rankings can be computed efficiently using minimum cost perfect matching. Diaconis and Graham [7] established that the Spearman footrule distance and the Kendall tau distance are within a factor of 2 of each other. This automatically implies that the footrule-optimal aggregation is a factor 2 approximation to the optimum using the Kendall tau distance as proposed by Kemeny. The Spearman rho distance is closely related to another quantity, the Spearman correlation coefficient, which is essentially

[1] All candidates are treated equally, i.e., if two candidates switch positions in every voter's ranking, then they must switch positions in the aggregate as well.

[2] If the voters are split into A and B, and both the aggregate of A and the aggregate of B prefer some candidate a to another candidate b, then the aggregate of $A \cup B$ should also prefer a to b.

[3] All voters are treated equally, i.e., the aggregate will not change if any two voters traded their rankings.

[4] Determining the Kemeny winner is complete for $\mathrm{P}_{\|}^{\mathrm{NP}}$, the class of sets solvable via parallel access to NP [10].

the Pearson correlation coefficient between the ranks of the candidates in the two permutations.

1.3. Comparison with related work. A few years ago, it was established that *median ranking* is a factor 6 approximation to the Kendall optimum [9]. It was shown in [6] that the Borda count method leads to a factor 5 approximation to the Kendall optimum as well (as noted in [6], we independently, around the same time, showed that the Borda count method leads to a constant-factor approximation to the Kendall optimum). It has been unknown whether either method yields a constant-factor approximation to the Spearman rho optimum as well (note that unlike the Spearman footrule distance, the Spearman rho distance is not bounded by a constant multiple of the Kendall tau distance). We show that the Borda, Copeland, and median aggregation methods all yield constant-factor approximations with respect to all three distance measures. Our methodology builds on [9] and extends it significantly to unify all proofs into a single framework. We have not optimized the factors of approximations, but have settled for clear proofs that establish constant-factor approximations. An approximation lower bound of 2 for the Borda and footrule methods was established in [14].

Since the Spearman footrule distance and the Kendall tau distance are within constant factors of each other, a constant factor approximation using the Spearman footrule distance automatically gives a constant factor approximation using the Kendall tau distance, and vice versa. So we actually have six distinct positive results (Borda, Copeland, and median voting methods with distances measured by the Spearman footrule or the Spearman rho distance). Of these six positive results, four are new (specifically all but the two we mentioned earlier), and for the two that are not new, we provide simplified proofs, in a uniform framework. We also obtain three new negative results about approximations for other voting methods.

2. Preliminaries.

2.1. Distances between permutations. Let S_n be the set of permutations on elements $[n] = \{1, \ldots, n\}$ and let V be any vector space. For a permutation $\sigma \in S_n$, we use $\sigma(i)$ to denote the *rank* of the element i. For example, in the permutation σ of $[3] = \{1, 2, 3\}$, where 3 is first, 1 is second, and 2 is third, we have $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$. It is convenient for us to represent a permutation by the vector of the ranks of its elements. In our example we would represent σ by $(2, 3, 1)$. It is important not to misinterpret this representation $(2, 3, 1)$ by thinking incorrectly that it means that 2 is first, 3 is second, and 1 is third.

There are several ways to define the distance between two permutations. Given $\sigma_1, \sigma_2 \in S_n$, the following are three popular notions of distances between them:

1. *Spearman footrule* distance: $F(\sigma_1, \sigma_2) = \sum_{i=1}^{n} |\sigma_1(i) - \sigma_2(i)|$.
2. *Kendall tau* distance: $K(\sigma_1, \sigma_2) = \sum_{i,j} A_{ij}$, where $A_{ij} = 1$ if $\sigma_1(i) < \sigma_1(j)$ and $\sigma_2(i) > \sigma_2(j)$, and $A_{ij} = 0$ otherwise. Intuitively, the Kendall tau distance is the number of inversions.
3. *Spearman rho* distance: $\sum_{i=1}^{n} (\sigma_1(i) - \sigma_2(i))^2$.

Let $L_p(u, v)$ denote the distance between two n-element vectors u and v as measure by the L_p metric: $L_p(u, v) = (\sum_{i=1}^{n} |u(i) - v(i)|^p)^{1/p}$. The footrule distance can be interpreted as the L_1 distance and the Spearman rho distance can be interpreted as the L_2^2 distance between vectors of the ranks of the elements. A function $d : V^2 \to \mathbb{R}$ is called a *distance function* if for every $x, y \in V$ we have (i) $d(x, y) \geq 0$, (ii) $d(x, y) = 0$ if and only if $x = y$, and (iii) $d(x, y) = d(y, x)$. The distance function d satisfies the *c-approximate triangle inequality* if for all $x, y, z \in V$, we have $d(x, z) \leq c \cdot (d(x, y) + d(y, z))$. Of course, the standard triangle inequality is the same as the 1-approximate triangle inequality. It is easy to see that $F(\cdot, \cdot)$ and $K(\cdot, \cdot)$ satisfy the triangle inequality. It is well known (and can easily be shown by the arithmetic-geometric mean inequality) that $L_2^2(\cdot, \cdot)$ satisfies the 2-approximate triangle inequality.

The following relates the footrule and Kendall tau distances to within constant factors of each other.

LEMMA 2.1 (Diaconis–Graham inequality [7]). *Assume $\sigma_1, \sigma_2 \in S_n$. Then $K(\sigma_1, \sigma_2) \leq F(\sigma_1, \sigma_2) \leq 2K(\sigma_1, \sigma_2)$.*

2.2. Approximate aggregation. We now define the notion of (approximate) aggregation.

DEFINITION 2.2 (approximate aggregation). *Let U and V be sets with $U \subseteq V$ and let $d : V^2 \to \mathbb{R}$ be a distance function. A function $h : U^k \to V$ is a (U, V, d, b)-*aggregation *function if for all $u_1, \ldots, u_k, u \in U$ we have $\sum_{\ell=1}^{k} d(h(u_1, \ldots, u_k), u_\ell) \leq b \cdot \sum_{\ell=1}^{k} d(u, u_\ell)$.*

Thus, $h(u_1, \ldots, u_k)$ is a member of V such that the sum of its distances from the u_ℓ's is within a factor of b of being as small as that for each member of U. This definition is very general and we will use it for several choices of U and V. We shall usually, but not always, have $U = V$. The familiar rank aggregation problem is to obtain an $(S_n, S_n, d, 1)$-aggregation function, where d is a distance function on permutations. An (S_n, S_n, d, c)-aggregation function for $c > 1$ is therefore a *c-approximate rank aggregation*. We say that an aggregation function is a *constant-factor approximation* if it is c-approximate for some constant c.

It is known that the footrule optimum (i.e., an $(S_n, S_n, F, 1)$-aggregation function) and the Spearman rho optimum can be obtained in polynomial time [8], whereas the problem of finding the Kendall optimum (i.e., an $(S_n, S_n, K, 1)$-aggregation function) is NP-hard [3]. Using Lemma 2.1, however, one can obtain an $(S_n, S_n, K, 2)$-aggregation function in polynomial time; by [13], one can obtain an $(S_n, S_n, K, (1+\epsilon))$-aggregation function in polynomial time, for every fixed $\epsilon > 0$.

2.3. Induced permutations. We now define the notion of a permutation induced by a vector of values.

DEFINITION 2.3 (induced permutation). *Given $\hat{\alpha} \in \mathbb{R}^n$, the* induced permutation *of $\hat{\alpha}$ is a permutation $\alpha \in S_n$ such that $\hat{\alpha}(i) < \hat{\alpha}(j) \implies \alpha(i) < \alpha(j)$ for all i, j. Note that ties may be broken arbitrarily.*

We note that all of our results hold however the ties are broken (that is, whichever choice of the induced permutation is made, when there is more than one option).

As an example, if $\hat{\alpha} = (2.5, 2.1, 7, 2.5)$, then, using our representation of permutations based on ranks of the elements, the induced permutation of $\hat{\alpha}$ can be taken to be either $(2, 1, 4, 3)$ or $(3, 1, 4, 2)$.

For clarity, we let $\text{ind} : \mathbb{R}^n \to S_n$ denote the function mapping a vector to its induced permutation. Thus, in Definition 2.3, we have $\alpha = \text{ind}(\hat{\alpha})$. We now show a simple yet crucial fact about induced permutations, namely, that $\hat{\alpha}$ is at least as close (in L_p^p for each $p \geq 1$) to $\text{ind}(\hat{\alpha})$ as it is to to any other permutation.

LEMMA 2.4 (induced permutations are optimal). *Assume that $p \geq 1$ and $\hat{\alpha} \in \mathbb{R}^n$. Let $\alpha = \text{ind}(\hat{\alpha})$, and let τ be an arbitrary permutation. Then $L_p^p(\alpha, \hat{\alpha}) \leq L_p^p(\tau, \hat{\alpha})$.*

Proof. Assume first that both α and τ are possible induced permutations of $\hat{\alpha}$. It is straightforward to verify that $L_p^p(\alpha,\hat{\alpha}) = L_p^p(\tau,\hat{\alpha})$. So assume that τ is not an induced permutation of $\hat{\alpha}$. Therefore, there is a pair (i,j) such that $\hat{\alpha}(i) < \hat{\alpha}(j)$ but $\tau(i) > \tau(j)$. By Definition 2.3, it follows that $\alpha(i) < \alpha(j)$.

We now show that $L_p^p(\alpha,\hat{\alpha}) \le L_p^p(\tau,\hat{\alpha})$. The proof is by an exchange argument. We will show that executing a swap of i and j cannot increase the distance to $\hat{\alpha}$. Since a sequence of such swaps leads from τ to α, this is sufficient to prove the result.

Let $\tilde{\tau}$ be obtained by swapping i and j in τ. So $\tilde{\tau}(i) = \tau(j)$ and $\tilde{\tau}(j) = \tau(i)$, while $\tilde{\tau}(\ell) = \tau(\ell)$ if $\ell \notin \{i,j\}$. We show that $L_p^p(\tilde{\tau},\hat{\alpha}) \le L_p^p(\tau,\hat{\alpha})$ or, equivalently, $L_p^p(\tilde{\tau},\hat{\alpha}) - L_p^p(\tau,\hat{\alpha}) \le 0$.

$$\begin{aligned}L_p^p(\tilde{\tau},\hat{\alpha}) - L_p^p(\tau,\hat{\alpha}) &= |\tilde{\tau}(i)-\hat{\alpha}(i)|^p + |\tilde{\tau}(j)-\hat{\alpha}(j)|^p - |\tau(i)-\hat{\alpha}(i)|^p - |\tau(j)-\hat{\alpha}(j)|^p \\ &= |\tau(j)-\hat{\alpha}(i)|^p + |\tau(i)-\hat{\alpha}(j)|^p - |\tau(i)-\hat{\alpha}(i)|^p - |\tau(j)-\hat{\alpha}(j)|^p.\end{aligned}$$

So we need only show that

$$0 \ge |\tau(j)-\hat{\alpha}(i)|^p + |\tau(i)-\hat{\alpha}(j)|^p - |\tau(i)-\hat{\alpha}(i)|^p - |\tau(j)-\hat{\alpha}(j)|^p. \tag{2.1}$$

Let $f(z) = |z-\hat{\alpha}(i)|^p - |z-\hat{\alpha}(j)|^p$. We now show that f is nondecreasing. Since $\hat{\alpha}(i) < \hat{\alpha}(j)$, we consider separately the region $z \le \hat{\alpha}(i)$, where $f(z) = (\hat{\alpha}(i)-z)^p - (\hat{\alpha}(j)-z)^p$; the region $\hat{\alpha}(i) \le z \le \hat{\alpha}(j)$, where $f(z) = (z-\hat{\alpha}(i))^p - (\hat{\alpha}(j)-z)^p$; and the region $z \ge \hat{\alpha}(j)$, where $f(z) = (z-\hat{\alpha}(i))^p - (z-\hat{\alpha}(j))^p$. By taking derivatives, it is straightforward to verify that f is nondecreasing in each of these regions, and so f is nondecreasing. Therefore, since $\tau(i) > \tau(j)$, we have

$$0 \ge f(\tau(j)) - f(\tau(i)) = |\tau(j)-\hat{\alpha}(i)|^p - |\tau(j)-\hat{\alpha}(j)|^p - |\tau(i)-\hat{\alpha}(i)|^p + |\tau(i)-\hat{\alpha}(j)|^p.$$

By rearranging terms, this shows that (2.1) holds, as desired. ☐

3. Approximate rank aggregation via approximate aggregation. In this section we show that if h is a constant-factor approximate aggregation function, then $\mathrm{ind} \circ h$ is also a constant-factor approximate (rank aggregation) function, and it satisfies a constant-factor approximate triangle inequality. (Recall that ind is the function mapping vectors to their induced permutations, and $\mathrm{ind} \circ h$ denotes an ordering induced by the values of h.) As stated earlier, we represent a permutation $\sigma \in S_n$ as an n-element vector of ranks.

THEOREM 3.1. *Assume that the vector space V in $\mathbb{R}^n$ contains our representation of members of S_n, and that U is either S_n or contains our representation of members of S_n. Let h be a (U,V,d,b)-aggregation function, where d is either L_1 or L_2^2. If d is L_1, then $\mathrm{ind} \circ h$ is an $(S_n,S_n,L_1,(2b+1))$-aggregation function. If d is L_2^2, then $\mathrm{ind} \circ h$ is an $(S_n,S_n,L_2^2,(6b+4))$-aggregation function.*

Proof. For input permutations $\sigma_1,\ldots,\sigma_k \in S_n$, let $\hat{\gamma} = h(\sigma_1,\ldots,\sigma_k)$ and $\gamma = \mathrm{ind}(\hat{\gamma})$. The main idea in the proof is to use $\hat{\gamma}$ as an intermediate quantity and to use the optimality of the induced permutation γ. Let $c = 1$ if d is L_1, and let $c = 2$ if d is L_2^2. Thus, we have chosen c so that d satisfies the c-approximate triangle inequality. Let π be an arbitrary member of S_n.

$$\begin{aligned}\sum_{\ell=1}^k d(\gamma,\sigma_\ell) &\le c\left(\sum_{\ell=1}^k d(\gamma,\hat{\gamma}) + \sum_{\ell=1}^k d(\hat{\gamma},\sigma_\ell)\right) && \because c\text{-approximate triangle inequality of } d \\ &\le c\left(\sum_{\ell=1}^k d(\pi,\hat{\gamma}) + \sum_{\ell=1}^k d(\hat{\gamma},\sigma_\ell)\right) && \because \text{ Lemma 2.4 with } \alpha=\gamma \text{ and } \tau=\pi\end{aligned}$$

$$\leq c^2 \sum_{\ell=1}^{k} d(\pi, \sigma_\ell) + (c^2 + c) \sum_{\ell=1}^{k} d(\hat{\gamma}, \sigma_\ell) \quad \because d(\pi, \hat{\gamma}) \leq c(d(\pi, \sigma_\ell) + d(\sigma_\ell, \hat{\gamma}))$$

$$\leq ((b+1)c^2 + bc) \sum_{\ell=1}^{k} d(\pi, \sigma_\ell). \quad \because h \text{ is a } (U, V, d, b)\text{-aggregation function.}$$

If we substitute c for its value (1 for L_1 and 2 for L_2^2) in $(b+1)c^2 + bc$, we get $2b+1$ for L_1, and $6b+4$ for L_2^2. The result then follows. ∎

The power of this theorem will become apparent in the next section, where we see that five well-known rank-aggregation problems can each be viewed in the same framework: aggregate a set of input permutations to get a vector in $\mathbb{R}^n$, then move that vector back to the set of permutations through the induced permutation.

4. Five applications. In this section we show five key applications of the generic approximate aggregation. In particular, we focus on the median, Borda, and Copeland voting methods (for both Spearman footrule and Spearman rho) and show that they are approximate rank aggregation methods.

4.1. Median voting. Median voting is defined in the following manner. Given $\sigma_1, \ldots, \sigma_k$, we define the *median rank* of an element to be the median of its rank in the given k permutations. More formally, let $h_{\mathrm{med}}(\sigma_1, \ldots, \sigma_k)$ be the n-dimensional vector whose ith coordinate is given by

$$h_{\mathrm{med}}(\sigma_1, \ldots, \sigma_k)(i) = \mathrm{median}\{\sigma_1(i), \ldots, \sigma_k(i)\}.$$

Median voting is then given by ordering the elements according to their median ranks, that is, ordering them by their values in $\mathrm{ind} \circ h_{\mathrm{med}}(\sigma_1, \ldots, \sigma_k)$. In this and in all of our voting schemes, ties are broken arbitrarily. In the case of the median with an even number k of elements, either the element $\frac{k}{2}$ or $\frac{k}{2}+1$ in the ordering is selected. In this subsection we show that median voting is a 3-approximation to the footrule optimum.

We first claim (in Lemma 4.1) that the median is the best function to minimize the L_1 distance.

LEMMA 4.1. *The function h_{med} is an $(\mathbb{R}^n, \mathbb{R}^n, L_1, 1)$-aggregation function. Thus, given $u_1, \ldots, u_k \in \mathbb{R}^n$, if $\hat{\mu} = h_{\mathrm{med}}(u_1, \ldots, u_k)$, then for every $u \in \mathbb{R}^n$, we have $\sum_{\ell=1}^{k} L_1(\hat{\mu}, u_\ell) \leq \sum_{\ell=1}^{k} L_1(u, u_\ell)$.*

Proof. For a set X of numbers, it is easy to see that

$$\mathrm{median}(X) = \arg\min_{\hat{x}} \sum_{x \in X} |\hat{x} - x|,$$

i.e., the median is the L_1 minimizer. Since $\hat{\mu}(i) = \mathrm{median}\{u_1(i), \ldots, u_k(i)\}$, we have

$$\sum_{\ell=1}^{k} L_1(\hat{\mu}, u_\ell) = \sum_{\ell=1}^{k} \sum_{i=1}^{n} |\hat{\mu}(i) - u_\ell(i)| \leq \sum_{\ell=1}^{k} \sum_{i=1}^{n} |u(i) - u_\ell(i)| = \sum_{\ell=1}^{k} L_1(u, u_\ell)$$

for every $u \in \mathbb{R}^n$. ∎

THEOREM 4.2. *Median voting is a 3-approximation to the footrule optimum.*

Proof. We apply Theorem 3.1 with $h = h_{\mathrm{med}}$ and $d = L_1$. From Lemma 4.1, we have $b = 1$. ∎

Theorem 4.2 says that $\mathrm{ind} \circ h_{\mathrm{med}}$ is an $(S_n, S_n, F, 3)$-aggregation function. Thus, given $\sigma_1, \ldots, \sigma_k \in S_n$, let $\mu = \mathrm{ind}(h_{\mathrm{med}}(\sigma_1, \ldots, \sigma_k))$. Then $\sum_{\ell=1}^k F(\mu, \sigma_\ell) \leq 3 \sum_{\ell=1}^k F(\pi, \sigma_\ell)$ for each $\pi \in S_n$.

Applying Lemma 2.1, we conclude that median voting is an $(S_n, S_n, K, 6)$-aggregation function, that is, a 6-approximation to the Kendall optimum. We next show that median voting is a constant-factor approximation to the Spearman rho optimum. We begin with a lemma.

LEMMA 4.3. *The function h_{med} is an $(S_n, \mathbb{R}^n, L_2^2, 2)$-aggregation function.*

Proof. We first claim the following. Assume that $a \leq x \leq b$. Then for each z, we have

$$(a-x)^2 + (b-x)^2 \leq 2(a-z)^2 + 2(b-z)^2. \tag{4.1}$$

To prove this, let f be the function where $f(z) = (a-z)^2 + (b-z)^2$. By calculus we can see that f attains its minimum value at $z = \frac{a+b}{2}$. Now f is monotone decreasing before $z = \frac{a+b}{2}$ and monotone increasing afterwards. So over the interval $a \leq x \leq b$, the left-hand side of (4.1) attains its maximum value at the endpoints a and b, and this maximum value is $(a-b)^2$. From what we said earlier, the right-hand side of (4.1) attains its minimum value at $z = \frac{a+b}{2}$, and this value is $(a-b)^2$. Since the maximum value of the left-hand side of (4.1) equals the minimum value of the right-hand side of (4.1), the inequality (4.1) holds.

Now, given $\sigma_1, \ldots, \sigma_k, \pi \in S_n$, let $\hat{v} = h_{\mathrm{med}}(\sigma_1, \ldots, \sigma_k)$. We must show that $\sum_\ell L_2^2(\hat{v}, \sigma_\ell) \leq 2 \sum_\ell L_2^2(\pi, \sigma_\ell)$. Fix i and without loss of generality, assume that $\sigma_1(i) \leq \sigma_2(i) \leq \cdots \leq \sigma_k(i)$. Suppose k is odd. We then have $\hat{v}(i) = \sigma_{(k+1)/2}(i)$, and hence

$$\begin{aligned}
&\sum_\ell (\hat{v}(i) - \sigma_\ell(i))^2 \\
&\quad = \left(\sum_{q=1}^{(k-1)/2} (\hat{v}(i) - \sigma_q(i))^2 + (\hat{v}(i) - \sigma_{k+1-q}(i))^2 \right) + (\hat{v}(i) - \sigma_{(k+1)/2}(i))^2 \\
&\quad = \left(\sum_{q=1}^{(k-1)/2} (\hat{v}(i) - \sigma_q(i))^2 + (\hat{v}(i) - \sigma_{k+1-q}(i))^2 \right) \\
&\quad \leq \left(2 \sum_{q=1}^{(k-1)/2} (\pi(i) - \sigma_q(i))^2 + (\pi(i) - \sigma_{k+1-q}(i))^2 \right) \\
&\quad \leq \left(2 \sum_{q=1}^{(k-1)/2} (\pi(i) - \sigma_q(i))^2 + (\pi(i) - \sigma_{k+1-q}(i))^2 \right) + 2(\pi(i) - \sigma_{(k+1)/2}(i))^2 \\
&\quad = 2 \sum_\ell (\pi(i) - \sigma_\ell(i))^2,
\end{aligned}$$

where the first inequality follows from (4.1), with the roles of a, b, x, z played by $\sigma_q(i), \sigma_{k+1-q}(i), \hat{v}(i), \pi(i)$, respectively. Summing over i establishes $\sum_\ell L_2^2(\hat{v}, \sigma_\ell) \leq 2 \sum_\ell L_2^2(\pi, \sigma_\ell)$. The proof when k is even is analogous. ∎

THEOREM 4.4. *Median voting is a* 16*-approximation to the Spearman rho optimum.*

Proof. We apply Theorem 3.1 with $h = h_{\text{med}}$ and $d = L_2^2$. From Lemma 4.3, we have $b = 2$. ∎

4.2. Borda voting. Borda voting is defined as follows. Let $\sigma_1, \ldots, \sigma_k$ be the given permutations. For each element, we compute its Borda count as the mean of its ranks in each of the given k permutations (equivalently, we could make use of the sum, rather than the median, of its ranks). *Borda voting* is then given by ordering the elements according to their Borda counts. Note that Borda voting is in the spirit of median voting, except that the mean is used instead of the median. In this section our main result is that Borda voting is a 5-approximation to the footrule optimum.

More formally, let h_{avg} be the n-dimensional vector whose ith coordinate is given by

$$h_{\text{avg}}(\sigma_1, \ldots, \sigma_k)(i) = \frac{1}{k}\sum_{\ell=1}^{k} \sigma_\ell(i).$$

The permutation produced by Borda voting is then $\operatorname{ind}(h_{\text{avg}}(\sigma_1, \ldots, \sigma_k))$.

LEMMA 4.5. *The function h_{avg} is an $(\mathbb{R}^n, \mathbb{R}^n, L_1, 2)$-aggregation function.*

Proof. Assume that $u_1, \ldots, u_k \in \mathbb{R}^n$. Let $\hat{\beta} = h_{\text{avg}}(u_1, \ldots, u_k)$ and $\hat{\mu} = h_{\text{med}}(u_1, \ldots, u_k)$. We will show

$$\sum_{\ell=1}^{k} L_1(\hat{\beta}, u_\ell) \le 2\sum_{\ell=1}^{k} L_1(\hat{\mu}, u_\ell);$$

the proof will then follow from Lemma 4.1. For every $i \in [n]$,

$$\begin{aligned}
\sum_\ell |\hat{\beta}(i) - u_\ell(i)| &= \sum_\ell \left|\left(\frac{1}{k}\sum_{\ell'} u_{\ell'}(i)\right) - u_\ell(i)\right| \qquad \because \text{definition of } \hat{\beta} \\
&= \frac{1}{k}\sum_\ell \left|\sum_{\ell'} (u_{\ell'}(i) - u_\ell(i))\right| \\
&\le \frac{1}{k}\sum_{\ell,\ell'} |u_{\ell'}(i) - u_\ell(i)| \qquad \because \text{triangle inequality} \\
&\le \frac{1}{k}\sum_{\ell,\ell'} (|u_{\ell'}(i) - \hat{\mu}(i)| + |\hat{\mu}(i) - u_\ell(i))|) \qquad \because \text{triangle inequality} \\
&= \frac{1}{k}\sum_{\ell,\ell'} |u_{\ell'}(i) - \hat{\mu}(i)| + \frac{1}{k}\sum_{\ell,\ell'} |\hat{\mu}(i) - u_\ell(i))| \\
&= \sum_{\ell'} |u_{\ell'}(i) - \hat{\mu}(i)| + \sum_\ell |\hat{\mu}(i) - u_\ell(i))| \\
&= 2\sum_\ell |\hat{\mu}(i) - u_\ell(i))|.
\end{aligned}$$

By summing the above inequality over $i \in [n]$, the proof is complete. ∎

THEOREM 4.6. *Borda voting is a 5-approximation to the footrule optimum.*

Proof. We apply Theorem 3.1 with $h = h_{\text{avg}}$ and $d = L_1$. From Lemma 4.5, we have $b = 2$. ∎

As before, applying Lemma 2.1, we conclude that Borda voting is an $(S_n, S_n, K, 10)$-aggregation function, that is, a 10-approximation to the Kendall optimum. For Spearman rho, it is well known that $\sum_{i=1}^{k}(a_i - x)^2$ is minimized when $x = \frac{1}{k}\sum_i a_i$. (This can also be seen easily by taking a derivative.) Notice that this immediately implies that h_{avg} is an $(\mathbb{R}^n, \mathbb{R}^n, L_2^2, 1)$-aggregation function. Hence, applying Theorem 3.1 (with $d = L_2^2$ and $b = 1$), we have the following.

THEOREM 4.7. *Borda voting is a* 10*-approximation to the Spearman rho optimum.*

4.3. Copeland voting. Before we give our results on Copeland voting, we present a new and surprising identity, that we make use of in our analysis of Copeland voting.

Define

$$w_{i,j}^{\pi,\sigma} = \begin{cases} |\pi(i) - \pi(j)| & \text{if } (\pi(i) < \pi(j) \wedge \sigma(i) > \sigma(j)) \text{ or } (\pi(i) > \pi(j) \wedge \sigma(i) < \sigma(j)), \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 4.8. *Assume that* π *and* σ *are each permutations on* $[n]$*. Then* $L_2^2(\pi, \sigma) = \sum_{i,j} w_{i,j}^{\pi,\sigma}$*, where* $w_{i,j}^{\pi,\sigma}$ *is as defined above.*

Proof. We begin by showing that we can assume without loss of generality that σ is the identity permutation (denoted 1). As a first step, we show that L_2^2 is right invariant, that is, that if π, σ, τ are all permutations on the same domain, then $L_2^2(\pi, \sigma) = L_2^2(\pi \circ \tau, \sigma \circ \tau)$. This is because $L_2^2(\pi, \sigma) = \sum_i(\pi(i) - \sigma(i))^2 = \sum_i(\pi(\tau(i)) - \sigma(\tau(i)))^2 = L_2^2(\pi \circ \tau, \sigma \circ \tau)$, where in the second equality we are doing a simple reordering of the domain. By letting τ be σ^{-1}, we obtain

$$L_2^2(\pi, \sigma) = L_2^2(\pi \circ \sigma^{-1}, 1). \tag{4.2}$$

By assuming that the theorem is proven when the second permutation is the identity permutation, we obtain

$$L_2^2(\pi \circ \sigma^{-1}, 1) = \sum_{i,j} w_{i,j}^{\pi\circ\sigma^{-1},1} = \sum_{i,j} w_{\sigma(i),\sigma(j)}^{\pi\circ\sigma^{-1},1}. \tag{4.3}$$

The second equality holds by doing a simple reordering of the domain, as before.

By letting the roles of π, σ, i, j in the definition of $w_{i,j}^{\pi,\sigma}$ be played by $\pi \circ \sigma^{-1}$, 1, $\sigma(i)$, $\sigma(j)$, respectively, we obtain after plugging these latter values into the definition that

$$w_{\sigma(i),\sigma(j)}^{\pi\circ\sigma^{-1},1} = w_{i,j}^{\pi,\sigma}. \tag{4.4}$$

By combining the equalities (4.2), (4.3), and (4.4), we obtain $L_2^2(\pi, \sigma) = \sum_{i,j} w_{i,j}^{\pi,\sigma}$, as desired.

Therefore, it suffices to show that for each permutation π, we have

$$L_2^2(\pi) = \sum_{i,j} w_{i,j}^{\pi}, \tag{4.5}$$

where $w_{i,j}^{\pi} = w_{i,j}^{\pi,1}$. Thus, $w_{i,j}^{\pi} = |\pi(i) - \pi(j)|$ if $(i < j) \wedge (\pi(i) > \pi(j))$ or if $(i > j) \wedge (\pi(i) < \pi(j))$, and $w_{i,j}^{\pi} = 0$ otherwise.

We have the following series of equalities, where the explanations appear in the paragraph following the equations.

$$\sum_{i,j} w^{\pi}_{i,j} = 2 \sum_{1\le i<j\le n} \max(\pi(i)-\pi(j),0) \tag{4.6}$$

$$= \sum_{1\le i<j\le n} (|\pi(i)-\pi(j)| + \pi(i)-\pi(j)) \tag{4.7}$$

$$= \sum_{1\le i<j\le n} |\pi(i)-\pi(j)| + \sum_{1\le i<j\le n} \pi(i) - \sum_{1\le i<j\le n} \pi(j) \tag{4.8}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} |\pi(i)-\pi(j)| + \sum_{i=1}^{n}(n-i)\pi(i) - \sum_{j=1}^{n}(j-1)\pi(j) \tag{4.9}$$

$$= \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} |k-l| + \sum_{i=1}^{n}(n-i)\pi(i) - \sum_{i=1}^{n}(i-1)\pi(i) \tag{4.10}$$

$$= \sum_{1\le k<l\le n} |k-l| + \sum_{i=1}^{n}(n-2i+1)\pi(i) \tag{4.11}$$

$$= \sum_{1\le k<l\le n} l - \sum_{1\le k<l\le n} k + \sum_{i=1}^{n}(n-2i+1)\pi(i) \tag{4.12}$$

$$= \sum_{l=1}^{n}(l-1)l - \sum_{k=1}^{n}(n-k)k + \sum_{i=1}^{n}(n-2i+1)\pi(i) \tag{4.13}$$

$$= \sum_{l=1}^{n} l^2 - \sum_{l=1}^{n} l - \sum_{k=1}^{n} nk + \sum_{k=1}^{n} k^2 + \sum_{i=1}^{n}(n-2i+1)\pi(i) \tag{4.14}$$

$$= \sum_{l=1}^{n} l^2 - \sum_{i=1}^{n} i - \sum_{i=1}^{n} ni + \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n}(n-2i+1)\pi(i) \tag{4.15}$$

$$= \sum_{i=1}^{n} \pi(i)^2 - (n+1)\sum_{i=1}^{n} \pi(i) + \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n}(n-2i+1)\pi(i) \tag{4.16}$$

$$= \sum_{i=1}^{n}(\pi(i)^2 + i^2 - 2i\pi(i)) \tag{4.17}$$

$$= \sum_{i=1}^{n}(\pi(i)-i)^2 \tag{4.18}$$

$$= L_2^2(\pi).$$

Explanations: Equation (4.6) holds, since it follows easily from the definition of $w^{\pi}_{i,j}$ that for every $i<j$, we have $w^{\pi}_{i,j} = w^{\pi}_{j,i} = \max(\pi(i)-\pi(j),0)$. Equation (4.7) holds because of the identity $2\max(x,0) = |x| + x$. Equation (4.8) is separating the three parts of the summation. Equation (4.9) uses the fact that $|\pi(i)-\pi(j)| = |\pi(j)-\pi(i)|$ and therefore this term is counted once when the summation is over all $i<j$ and twice when the sum is over all i,j. Equation (4.10) uses the fact that summing over all $i = 1,\dots,n$ is equivalent to summing over all $k = 1,\dots,n$ and setting $i = \pi^{-1}(k)$ (and similarly for j). Equation (4.11) follows from the fact that each $|k-l|$ is counted once in summation over $k<l$ and twice in summation over all k,l. Equation (4.12) uses the fact that when $k<l$, we have $|k-l| = l-k$, and separates the summation into two summations. Equation (4.13) holds, since there are $l-1$ choices for k and $n-k$ choices for l with $1\le k<l\le n$. Equation (4.14) separates the summations. In (4.15)

we rename the variable in three of the summations. Equation (4.16) uses the fact that summing l^2 over $l = 1, \ldots, n$ is equivalent to summing $\pi(i)^2$ over $i = 1, \ldots, n$ (and similarly, for summing i). In (4.17), we merge the summations into one and cancel the $(n+1)\pi(i)$ term.

In Appendix A, we give an alternative proof of (4.5) by induction. ∎

Note that in Theorem 4.8, the term $w_{i,j}^{\pi,\sigma}$ is counted twice in the sum: once as $w_{i,j}^{\pi,\sigma}$ and once as $w_{j,i}^{\pi,\sigma}$.

Before defining Copeland voting, we first need a few preliminaries. For the Kendall tau distance, define the *all-pairs function* $\psi : \mathbb{R}^n \to \{0,1\}^{n^2}$ as follows. Given $v \in \mathbb{R}^n$, let $\psi(v)$ be the *all-pairs vector* indexed by $i \in [n], j \in [n]$ with $\psi(v)(i,j) = 1$ if $v(i) \le v(j)$ and 0 otherwise. Then we see that the Kendall tau distance between permutations σ_1, σ_2 is simply half of the L_1 distance between $\psi(\sigma_1)$ and $\psi(\sigma_2)$, that is,

$$K(\sigma_1, \sigma_2) = \frac{1}{2} L_1(\psi(\sigma_1), \psi(\sigma_2)). \tag{4.19}$$

Given an all-pairs vector $\hat{v}$, we define the mapping $\phi : \mathbb{R}^{n^2} \to \mathbb{R}^n$ as $\phi(\hat{v})(i) = \sum_j \hat{v}(j,i)$. Note that $\phi(\psi(\sigma)) = \sigma$ for all $\sigma \in S_n$.

LEMMA 4.9. *For $\hat{u}, \hat{v} \in \mathbb{R}^{n^2}$, we have $L_1(\phi(\hat{u}), \phi(\hat{v})) \le L_1(\hat{u}, \hat{v})$.*

Proof. $L_1(\phi(\hat{u}), \phi(\hat{v})) = \sum_i |\sum_j \hat{u}(j,i) - \sum_j \hat{v}(j,i)| \le \sum_i \sum_j |\hat{u}(j,i) - \hat{v}(j,i)| = L_1(\hat{u}, \hat{v})$. ∎

We now define the Copeland rank aggregation method as follows. Given permutations $\sigma_1, \ldots, \sigma_k$, let $\hat{\kappa} \in \{0,1\}^{n^2}$ be the median vector of $\psi(\sigma_1), \ldots, \psi(\sigma_k)$, that is, for each i, j,

$$\hat{\kappa}(i,j) = \text{median}_\ell\{\psi(\sigma_\ell)(i,j)\},$$

that is, $\hat{\kappa} = h_{\text{med}}(\psi(\sigma_1), \ldots, \psi(\sigma_k))$. The final ranking κ is $\text{ind}(\phi(\hat{\kappa}))$. Let $h_{\text{cop}} = \phi \circ h_{\text{med}} \circ \psi$, where we take $\psi(u_1, \ldots, u_k)$ to be $(\psi(u_1), \ldots, \psi(u_k))$. The *Copeland voting* function is then $\text{ind} \circ h_{\text{cop}}$. Intuitively, we say that i has a pairwise victory over j if i is ranked higher (better) than j by more than half the voters; the final Copeland ranking is on the basis of the number of pairwise victories.

THEOREM 4.10. *Copeland voting is a 4-approximation to the Kendall optimum.*

Proof. Given $\sigma_1, \ldots, \sigma_k \in S_n$, let $\hat{\kappa}$ and κ be as defined earlier. Assume that $\pi \in S_n$. Then

$$\begin{aligned}
\sum_\ell K(\sigma_\ell, \kappa) &\le \sum_\ell F(\sigma_\ell, \kappa) && \because \text{Lemma 2.1} \\
&\le \sum_\ell (L_1(\sigma_\ell, \phi(\hat{\kappa})) + L_1(\phi(\hat{\kappa}), \kappa)) && \because \text{triangle inequality} \\
&\le 2 \sum_\ell L_1(\sigma_\ell, \phi(\hat{\kappa})) && \because \text{Lemma 2.4, } \kappa = \text{ind}(\phi(\hat{\kappa})) \\
&\le 2 \sum_\ell L_1(\psi(\sigma_\ell), \hat{\kappa}) && \because \text{Lemma 4.9, } \sigma_\ell = \phi(\psi(\sigma_\ell)) \\
&\le 2 \sum_\ell L_1(\psi(\sigma_\ell), \psi(\pi)) && \because \text{Lemma 4.1} \\
&= 4 \sum_\ell K(\sigma_\ell, \pi),
\end{aligned}$$

where the equality holds by (4.19). ∎

This immediately shows an 8-approximation to the footrule optimum. Finally, we show that the Copeland method is a constant-factor approximation even to the Spearman rho optimum. Unlike the cases of median and Borda voting, this turns out to be not as easy.

LEMMA 4.11. *The function h_{cop} is an $(S_n, \mathbb{R}^n, L_2^2, 10)$-aggregation function.*

Proof. Assume that $\sigma_1, \ldots, \sigma_k, \pi \in S_n$. Define the weight $w_{ij\ell}$ of a triple i, j, ℓ, as follows:

$$w_{ij\ell} = \begin{cases} |\pi(i) - \pi(j)| & \text{if } (\pi(i) < \pi(j) \wedge \sigma_\ell(i) > \sigma_\ell(j)) \text{ or } (\pi(i) > \pi(j) \wedge \sigma_\ell(i) < \sigma_\ell(j), \\ 0 & \text{otherwise.} \end{cases}$$

This turns out to be a key intermediate quantity. Let $u = h_{\text{cop}}(\sigma_1, \ldots, \sigma_k)$. We now show that

$$kL_2^2(\pi, u) \leq 4 \sum_{i,j,\ell} w_{ij\ell}. \tag{4.20}$$

For every i, let $t_i = |\pi(i) - u(i)|$. To show (4.20), we need only prove that

$$\sum_i kt_i^2/4 \leq \sum_{i,j,\ell} w_{ij\ell}. \tag{4.21}$$

We now temporarily hold i fixed (our proof will show that $kt_i^2/4 \leq \sum_{j,\ell} w_{ij\ell}$). Recall that

$$u(i) = \sum_j \text{median}\{\psi(\sigma_1)(j, i), \ldots, \psi(\sigma_k)(j, i)\}. \tag{4.22}$$

Assume that $\pi(i) \leq u(i)$ (the proof for the opposite case is similar).

Denote $\pi(i)$ by s. Since $t_i = |\pi(i) - u(i)|$ and $\pi(i) \leq u(i)$, it follows that $u(i) = t_i + s$. Let $J_i = \{j \mid \text{median}\{\psi(\sigma_1)(j, i), \ldots, \psi(\sigma_k)(j, i)\} = 1\}$. Since each $\psi(\sigma_\ell)(j, i)$ has value 0 or 1, also $\text{median}\{\psi(\sigma_1)(j, i), \ldots, \psi(\sigma_k)(j, i)\}$ has value either 0 or 1. By (4.22), it therefore follows that $u(i) = |J_i|$. Hence, $|J_i| = t_i + s$. Now the number of j such that $\pi(j) \leq \pi(i)$ is exactly $\pi(i)$, that is, s. So the number of members j of J_i where $\pi(i) < \pi(j)$ is at least $|J_i| - s = t_i$.

Let J_i' consist of those members j of J_i where $\pi(i) < \pi(j)$. Note that $i \notin J_i'$. We just showed that $|J_i'| \geq t_i$. Now for each j in J_i', let $A_j = \{\ell \mid \psi(\sigma_\ell)(j, i) = 1\}$. Then $|A_j| \geq k/2$, or else we would not have $\text{median}\{\psi(\sigma_1)(j, i), \ldots, \psi(\sigma_k)(j, i)\} = 1$. Now $\psi(\sigma_\ell)(j, i) = 1$ means that $\sigma_\ell(i) \geq \sigma_\ell(j)$, which implies that $\sigma_\ell(i) > \sigma_\ell(j)$, because $j \neq i$.

Putting this all together, we see that the set of pairs (j, ℓ) such that $j \in J_i'$ and $\ell \in A_j$ has cardinality at least $t_i k/2$, and for each such pair (j, ℓ), we have $\pi(i) < \pi(j)$ and $\sigma_\ell(i) > \sigma_\ell(j)$. Let J_i'' be the set of pairs (j, ℓ) such that $\pi(i) < \pi(j)$ and $\sigma_\ell(i) > \sigma_\ell(j)$. We just showed that $|J_i''| \geq t_i k/2$. Let $T = |J_i''| \geq t_i k/2$, and let $C_{q,i} = |\{(j, \ell) \in J_i'' : \ |\pi(i) - \pi(j)| = q\}|$.

We then have

$$\sum_{i,j,\ell} w_{ij\ell} = 2\sum_i \sum_{(j,\ell)\in J_i''} w_{ij\ell} = 2\sum_i \sum_{(j,\ell)\in J_i''} |\pi(i)-\pi(j)|$$
$$= 2\sum_i \sum_{q>0} q|\{(j,\ell)\in J_i'' : |\pi(i)-\pi(j)| = q\}| = 2\sum_i\sum_{q>0} qC_{q,i}. \tag{4.23}$$

The first equality holds, since $w_{ij\ell}$ double counts for each pair i,j (the second time when we reverse the roles of i and j), and since $w_{ij\ell}=0$ when $(j,\ell)\notin J_i''$. To prove (4.21), we need only show that $\sum_{q>0} qC_{q,i} \geq kt_i^2/8$. Since J_i'' is the disjoint union of the sets $C_{q,i}$ for $q>0$, it follows that $\sum_{q>0} C_{q,i} = T$, and hence $\sum_{q>0} C_{q,i} \geq T$. When we consider pairs (j,ℓ) in J_i'', there is just one j such that $\pi(j)=\pi(i)+1$, and there are at most k choices of ℓ. Therefore, there are at most k pairs (j,ℓ) in J_i'' such that $\pi(j)=\pi(i)+1$, so $C_{1,i}\leq k$; this implies that $\sum_{q>1} C_{q,i} \geq T-k$. Similarly, there are at most k pairs (j,ℓ) in J_i'' such that $\pi(j)=\pi(i)+2$, so we know that $C_{2,i}\leq k$; this implies that $\sum_{q>2} C_{q,i}\geq T-2k$. If we continue this argument, we see that $\sum_{q>a} C_{q,i} \geq T-ak$.

Summing these inequalities from $q=1$ up to $q=\lceil T/k\rceil$, we obtain

$$\sum_{q>0} qC_{q,i} \geq \sum_{q=1}^{\lceil T/k\rceil} qC_{q,i} \geq \sum_{a=0}^{\lceil T/k\rceil -1}\ \sum_{q>a} C_{q,i} \geq \sum_{a=0}^{\lceil T/k\rceil -1} (T-ak). \tag{4.24}$$

The first inequality holds since each summand $qC_{q,i}$ is nonnegative, and every summand of the second sum is a summand of the first sum. The final sum in (4.24) equals the following, where $\theta = \lceil T/k\rceil - T/k$ (so $0\leq\theta<1$):

$$\lceil T/k\rceil T - \frac{k}{2}(\lceil T/k\rceil -1)\lceil T/k\rceil = \left(\frac{T}{k}+\theta\right)T - \frac{k}{2}\left(\frac{T}{k}+\theta-1\right)\left(\frac{T}{k}+\theta\right).$$

The right-hand side equals

$$\frac{T^2}{k}\left(1-\frac{1}{2}\right) + T\left(\theta - \frac{\theta}{2} + \frac{1-\theta}{2}\right) + \frac{k}{2}\theta(1-\theta) = \frac{T^2}{2k} + \frac{T}{2} + \frac{k}{2}\theta(1-\theta) \geq \frac{T^2}{2k} \geq kt_i^2/8, \tag{4.25}$$

since $T\geq t_ik/2$. This was to be shown. This concludes the proof of (4.20).

If we let the role of σ in Theorem 4.8 be played by σ_ℓ, and thereby replace $w_{i,j}^{\pi,\sigma}$ in Theorem 4.8 by $w_{ij\ell}$, we obtain $L_2^2(\pi,\sigma_\ell) = \sum_{i,j} w_{ij\ell}$. Therefore,

$$\sum_\ell L_2^2(\pi,\sigma_\ell) = \sum_{i,j,\ell} w_{ij\ell}. \tag{4.26}$$

Finally, we have

$$\begin{aligned}
&\sum_\ell L_2^2(u,\sigma_\ell)\\
&\quad \leq 2\sum_\ell L_2^2(u,\pi) + 2\sum_\ell L_2^2(\pi,\sigma_\ell) \qquad \because \text{2-approximate triangle inequality}\\
&\quad = 2kL_2^2(u,\pi) + 2\sum_\ell L_2^2(\pi,\sigma_\ell)\\
&\quad \leq 8\sum_{i,j,\ell} w_{ij\ell} + 2\sum_\ell L_2^2(\pi,\sigma_\ell) \qquad \because (4.20)\\
&\quad = 10\sum_\ell L_2^2(\pi,\sigma_\ell). \qquad \because (4.26)
\end{aligned}$$

∎

THEOREM 4.12. *Copeland voting is a* 64*-approximation to the Spearman rho optimum.*

Proof. We apply Theorem 3.1 with $h = h_{\text{cop}}$ and $d = L_2^2$. From Lemma 4.11, we have $b = 10$. ◻

5. Unbounded rank aggregation methods. In this section we investigate the properties of three popular voting schemes, namely, plurality voting, Simpson–Kramer minmax voting, and a version of STV. We show that none of these schemes are a constant-factor approximation to the footrule optimum. While minmax obtains a total ordering of the candidates, we have to extend both plurality voting and STV in a natural manner to obtain a total ordering of the candidates (instead of identifying just the winner).

5.1. Plurality voting. *Plurality voting* is defined as follows. For each candidate i, let v_i be an n-dimensional vector, where n is the number of candidates and where $v_i(j)$ is the number of jth place votes that candidate i receives for $j = 1, \ldots, n$. Then candidate i defeats candidate i' if v_i precedes $v_{i'}$ lexicographically. In particular, if candidate i has strictly more first-place votes than candidate i', then candidate i defeats candidate i'. Note that plurality voting can be viewed as a modification of the Borda voting scheme, where a candidate receives 2^{n-m} points for each mth-place vote.

THEOREM 5.1. *Plurality voting is not a constant-factor approximation to the footrule optimum.*

Proof. We now give the scenario for our counterexample. We shall take the candidates to be $\{1, \ldots, n\}$, and the voters to be $\{1, \ldots, k\}$. We shall take the number n of candidates to be much bigger than the number k of voters. Voters 1 and 2 have the identity permutation, where they give their ith place vote to candidate i for $1 \le i \le n$. Voter ℓ gives his first-place vote to candidate $\ell - 1$ for $3 \le \ell \le k$, and his last-place vote to candidate 1. In particular, candidate 1 receives two first-place votes (from voters 1 and 2), and no other candidate receives more than one first-place vote. Hence, candidate 1 is the overall winner, even though $k - 2$ of the voters give him their last-place vote. For $3 \le \ell \le k$, voter ℓ gives his remaining votes (other than his first-place vote and his last-place vote) by ordering the remaining candidates numerically (preferring candidate i to candidate i' if $i < i'$). For example, voter 4, who is committed to giving candidate 3 his first-place vote and candidate 1 his last place vote, orders the candidates as $3, 2, 4, 5, 6 \ldots, n-1, n, 1$. We let σ_ℓ represent the permutation associated with voter ℓ for $1 \le \ell \le k$.

Let α be the identity permutation, which orders the candidates as $1, 2, \ldots, n$. It is easy to verify that α gives the result of plurality voting (although all we need is that candidate 1 comes out first under plurality voting). Let β be the permutation that orders the candidates $2, 3, \ldots, n-1, n, 1$.

The footrule cost associated with α is $\sum_{\ell=1}^{k} F(\alpha, \sigma_\ell)$. Now $F(\alpha, \sigma_\ell) \ge n - 1$ for $3 \le \ell \le k$, since $n - 1$ is the penalty caused by candidate 1 alone, for voters ℓ with $3 \le \ell \le k$. So the footrule cost $\sum_{\ell=1}^{k} F(\alpha, \sigma_\ell)$ is at least $(k-2)(n-1)$.

The footrule cost associated with β is $\sum_{\ell=1}^{k} F(\beta, \sigma_\ell)$. Of this cost, the total penalty caused by candidate 1 is $2(n-1)$ (all caused by voters 1 and 2). Each of the candidates $2, \ldots, k$ is somewhere in the first k positions for every voter. Hence, for each choice of one of the k voters and choice of one of these $k-1$ candidates, the penalty is less than k, and so the total penalty caused by all of the k voters with all of these $k-1$ candidates is less than k^3. Each of the remaining $n-k$ candidates ℓ with

$k+1 \leq \ell \leq n$ is in position ℓ for voters 1 and 2, in position $\ell-1$ for the remaining $k-2$ voters, and in position $\ell-1$ for β. Therefore, the total penalty caused by each of these $n-k$ candidates is 2 for a total penalty associated with these $n-k$ candidates of $2(n-k)$. Hence, the footrule cost $\sum_{\ell=1}^{k} F(\beta, \sigma_\ell)$ associated with β is at most $2(n-1)+k^3+2(n-k) = 4n+k^3-2k-2$.

The ratio $\sum_{\ell=1}^{k} F(\alpha, \sigma_\ell) / \sum_{\ell=1}^{k} F(\beta, \sigma_\ell)$ of the footrule cost of α to the footrule cost of β is therefore at least $(k-2)(n-1)/(4n+k^3-2k-2)$. If we hold k fixed and let n get arbitrarily large, this ratio is asymptotic to $(k-2)/4$. Since k is arbitrary, this asymptotic value can get arbitrarily large. Hence, plurality voting does not give a constant-factor approximation to the footrule optimum, since the cost of α is not within a constant factor of the cost of the solution β over all choices of the parameters k and n. ∎

5.2. Simpson–Kramer minmax voting. *Simpson–Kramer minmax voting* is defined as follows. Given permutations $\sigma_1, \ldots, \sigma_k$, define losses$(i,j)$ for a pair i,j of candidates to be the number of voters ℓ for which σ_ℓ ranks i lower (worse) than j. Further, define biggestLoss$(i) = \max_j\{\text{losses}(i,j)\}$. Then the induced ordering from minmax ranks the candidates in increasing order of biggestLoss$(\cdot)$, with the candidate having the smallest biggestLoss$(\cdot)$ winning.

THEOREM 5.2. *The induced ordering from minmax voting does not yield a constant-factor approximation to the footrule optimum.*

Proof. We now give a scenario for our counterexample. Consider the following set of votes over n candidates, where $N > n$ will be set later. For convenience, we assume that n is even.

- A total of N voters ranking candidates in order $1, 2, \ldots, n$; call this ranking σ_0.
- A total of n voters ranking candidates in order $1, n, n-1, \ldots, 2$; call this ranking σ_1.
- For each $i > 2$, one voter ranking candidates in order $i, i+1, \ldots, n, 1, i-1, i-2, \ldots, 2$; call this ranking σ_i.

Thus, the number k of voters is $N+2n-2$. First, note that candidate 1 beats candidate i exactly $N+n+(n-i)$ times for each $i > 1$. Second, note that for fixed i, j with $j \neq 1$, candidate j beats candidate i less than $N+n$ times: indeed, if $i < j$, then candidate j beats i at most $2n-2$ times, and if $i > j$, then candidate j beats i at most $N+n-2$ times.

Since also each $i \neq 1$ is beaten by 1 at least $N+n$ times, it follows that the ranking produced by minmax voting is determined by the number of times each i is beaten by 1, and this ranking is given by the permutation σ_1, So the ranking produced by minmax voting is σ_1. The total footrule distance to the input permutations from σ_1 is more than $N \cdot F(\sigma_0, \sigma_1) = N \cdot n(n-2)/2$. This equality is why we assumed that n is even.

On the other hand, the footrule distance of the inputs from σ_0 is given by

$$N \cdot F(\sigma_0, \sigma_0) + n \cdot F(\sigma_1, \sigma_0) + \sum_{i=3}^{n} F(\sigma_i, \sigma_0) \leq 2n^3,$$

where the final inequality holds because $F(\sigma_1, \sigma_0) = n(n-2)/2 < n^2$ and $F(\sigma_i, \sigma_0) < n^2$. Thus, as N grows, we see that the ratio of the minmax ranking distance to the distance for σ_0 (which is no better than optimal) is unbounded. ∎

5.3. STV. There are several variants of STV; we now describe one such variant. Given voter rankings $\sigma_1, \ldots, \sigma_k$, define the *loser* to be the candidate i with the fewest first-place votes, that is, loser $= \arg\min_i \{|\{\ell \mid \sigma_\ell(i) = 1\}|\}$. This loser is then removed from each ranking to produce new rankings, $\sigma'_1, \ldots, \sigma'_k$. (In this way, rankings that vote for losers can *transfer* their vote to someone who might win.) We do not describe how to deal with ties for the biggest loser; in our example, there will be no ties. The process is then repeated on the set of smaller rankings until all candidates but one have been removed. The remaining candidate is declared the winner. A natural way to induce a total ordering out of STV is put the first loser in last place, the second loser in second-to-last place, and so on. We take this as the induced ordering of the candidates.

THEOREM 5.3. *The induced ordering from STV does not yield a constant-factor approximation to the footrule optimum.*

Proof. We construct the following scenario for the counterexample. Consider the following set of rankings, where N will be set later. Again, we assume that n is even.

- A total of N voters ranking candidates in the order $1, 2, \ldots, n$; call this ranking σ_1.
- For each $i > 1$, a total of i voters ranking candidates in the order $i, 1, 2, \ldots, i-1, i+1, \ldots, n$; call this ranking σ_i.

Thus, the number k of voters is $N + \sum_{i=2}^{n} i$.

Assume that $N > n$. Now, consider STV. A candidate $i > 1$ has exactly i first-place votes. This remains true even after other candidates are removed. So the induced ranking τ ranks the candidates in the order $1, n, n-1, \ldots, 2$. Note that the total footrule distance of the permutations from τ is more than $N \cdot F(\sigma_1, \tau) = N \cdot n(n-2)/2$.

However, the total footrule distance from σ_1 is

$$N \cdot F(\sigma_1, \sigma_1) + \sum_{i=2}^{n} iF(\sigma_i, \sigma_1) \leq n^3.$$

Thus, as N grows, the ratio of the distance for the STV ranking versus the distance for the σ_1 ranking is unbounded. ∎

6. Conclusions. In this paper we showed that positional voting methods such as Borda counting, median ranking, and Copeland ranking are all constant-factor approximations with respect to the Kendall tau, Spearman footrule, and Spearman rho distance measures. We established this result by developing a general framework for reasoning about such methods. In contrast, we also show that natural extensions of each of plurality voting, Simpson–Kramer minmax voting, and STV are not constant-factor approximations. Our work thus offers a new perspective on a large class of voting methods. The literature on voting is nearly limitless. It will be interesting to bring the extensive set of prevalent voting methods into our framework and study their approximation quality with respect to the Kemeny optimality criterion.

Appendix A. Alternative proof of (4.5). Let us say that (i, j) is an *inversion in* π if either $i < j$ and $\pi(i) > \pi(j)$ or $i > j$ and $\pi(i) < \pi(j)$. If $(k, k+1)$ is an inversion in π, then the *adjacent swap* of k with $k+1$ is the permutation π' where $\pi'(k) = \pi(k+1)$, $\pi'(k+1) = \pi(k)$, and $\pi'(i) = \pi(i)$ for $i \notin \{k, k+1\}$. It is well known that from an arbitrary permutation on $[n]$, there is a sequence of adjacent swaps that leads to the identity permutation. To prove (4.5), we shall show that the changes in the left-hand side and right-hand side of (4.5) after an adjacent swap are identical. Since the left-hand side and right-hand side of (4.5) are equal (and, in fact, each equals 0) when π is the identity permutation, this proves that (4.5) holds for arbitrary π.

We first analyze the difference between the left-hand side of (4.5) for π' and π. Since all the terms other than k and $k+1$ cancel out in the left-hand side, we have

$$L_2^2(\pi')-L_2^2(\pi) = (\pi'(k)-k)^2+(\pi'(k+1)-(k+1))^2-(\pi(k)-k)^2-(\pi(k+1)-(k+1))^2.$$

Focusing on the right-hand side of the preceding equation, if we replace $\pi'(k)$ by $\pi(k+1)$ and replace $\pi'(k+1)$ by $\pi(k)$, we obtain

$$(\pi(k+1)-k)^2+(\pi(k)-(k+1))^2-(\pi(k)-k)^2-(\pi(k+1)-(k+1))^2.$$

If we now apply $a^2-b^2=(a+b)(a-b)$ to the first and the third terms and also to the second and the fourth terms, we obtain

$$(\pi(k+1)+\pi(k)-2k)(\pi(k+1)-\pi(k))+(\pi(k)+\pi(k+1)-2(k+1))(\pi(k)-\pi(k+1)).$$

If we now factor out $\pi(k)-\pi(k+1)$ from the two products of sums of terms, we see that the previous expression is simply

$$-2(\pi(k)-\pi(k+1)).$$

We now analyze the difference in the right-hand side of (4.5) due to this adjacent swap. Consider a generic term $w_{i,j}=w^{\pi}_{i,j}$ on the right-hand side corresponding to π. Let $w'_{i,j}=w^{\pi'}_{i,j}$ be the corresponding term for π'. We will study $\sum_{i,j} w'_{i,j}-\sum_{i,j} w_{i,j}$.

There are three disjoint cases to consider:

(i) $i,j \notin \{k,k+1\}$. In this case, $w_{i,j}=w'_{i,j}$ and hence the right-hand side difference is zero.

(ii) $\{i,j\}=\{k,k+1\}$. Suppose $i=k$ and $j=k+1$. In this case, $w'_{k,k+1}=0$ since the pair $(k,k+1)$ is no longer an inversion in π'. However, $w_{k,k+1}=(\pi(k)-\pi(k+1))$ since $(k,k+1)$ is an inversion in π. Likewise, if $i=k+1$ and $j=k$, then we have $w_{k+1,k}=(\pi(k)-\pi(k+1))$. Hence, the total right-hand side difference is $-2(\pi(k)-\pi(k+1))$.

(iii) Exactly one of i and j is in $\{k,k+1\}$. Assume without loss of generality that $i\in\{k,k+1\}$ and $j\notin\{k,k+1\}$. We shall show that $w_{j,k}=w'_{j,k+1}$ and $w_{j,k+1}=w'_{j,k}$. Hence, $w_{j,k}+w_{j,k+1}=w'_{j,k}+w'_{j,k+1}$, and so the right-hand side difference is zero.

We begin by showing that $w_{j,k}=w'_{j,k+1}$. There are two cases, depending on whether or not (j,k) is an inversion in π. Assume first that (j,k) is not an inversion in π. If $j<k$ and $\pi(j)<\pi(k)$, then $j<k+1$ and $\pi'(j)=\pi(j)<\pi(k)=\pi'(k+1)$ and, hence, $(j,k+1)$ is not an inversion in π'. Therefore, $w_{j,k}=0=w'_{j,k+1}$. If $j>k$ and $\pi(j)>\pi(k)$, then $j>k+1$ (since $j\neq k$) and $\pi'(j)=\pi(j)>\pi(k)=\pi'(k+1)$ and, hence, $(j,k+1)$ is not an inversion in π'. So $w_{j,k}=0=w'_{j,k+1}$.

Assume now that (j,k) is an inversion in π. If $j<k$ and $\pi(j)>\pi(k)$, then $j<k+1$ and $\pi'(j)=\pi(j)>\pi(k)=\pi'(k+1)$ and, hence, $(j,k+1)$ is an inversion in π'. If $j>k$ and $\pi(j)<\pi(k)$, then $j>k+1$ (since $j\neq k+1$) and $\pi'(j)=\pi(j)<\pi(k)=\pi'(k+1)$ and, hence, $(j,k+1)$ is an inversion in π'. We just showed that if (j,k) is an inversion in π, then $(j,k+1)$ is an inversion in π'. In these cases, $w_{j,k}=|\pi(j)-\pi(k)|=|\pi'(j)-\pi'(k+1)|=w'_{j,k+1}$, as desired.

We now show that $w_{j,k+1}=w'_{j,k}$. There are two cases, depending on whether or not $(j,k+1)$ is an inversion in π. Assume first that $(j,k+1)$ is not an inversion in π. If

$j < k+1$ and $\pi(j) < \pi(k+1)$, then $j < k$ (since $j \neq k$) and $\pi'(j) = \pi(j) < \pi(k+1) = \pi'(k)$ and, hence, (j,k) is not an inversion in π'. Therefore, $w_{j,k+1} = 0 = w'_{j,k}$. If $j > k+1$ and $\pi(j) > \pi(k+1)$, then $j > k$ and $\pi'(j) = \pi(j) > \pi(k+1) = \pi'(k)$ and, hence, (j,k) is not an inversion in π'. So $w_{j,k+1} = 0 = w'_{j,k}$.

Assume now that $(j,k+1)$ is an inversion in π. If $j < k+1$ and $\pi(j) > \pi(k+1)$, then $j < k$ (since $j \neq k$) and $\pi'(j) = \pi(j) > \pi(k+1) = \pi'(k)$ and, hence, (j,k) is an inversion in π'. If $j > k+1$ and $\pi(j) < \pi(k+1)$, then $j > k$ and $\pi'(j) = \pi(j) < \pi(k+1) = \pi'(k)$ and, hence, (j,k) is an inversion in π'. We just showed that if $(j,k+1)$ is an inversion in π, then (j,k) is an inversion in π'. In these cases, $w_{j,k+1} = |\pi(j) - \pi(k+1)| = |\pi'(j) - \pi'(k)| = w'_{j,k}$, as desired.

We have shown that the change in the left-hand side and right-hand side of (4.5) after an adjacent swap of k with $k+1$ is identical (in both cases it is $-2(\pi(k) - \pi(k+1))$. This was to be shown.

Acknowledgments. We ran Theorem 4.8 by Ronald Graham, Peter Winkler, and Richard Karp to see if they had ever seen this surprising identity before (they hadn't). We thank them for their input.

REFERENCES

[1] N. Ailon, M. Charikar, and A. Newman, *Aggregating inconsistent information: Ranking and clustering*, J. ACM, 55 (2008), 23.

[2] K. J. Arrow, *A difficulty in the concept of social welfare*, J. Political Econ., 58 (1950), pp. 328–346.

[3] J. J. Bartholdi, C. A. Tovey, and M. A. Trick, *Voting schemes for which it can be difficult to tell who won the election*, Social Choice Wel., 6 (1989), pp. 157–165.

[4] J. C. Borda, *Mémoire sur les élections au scrutin*, Hist. Acad. Roy. Sci., 1781.

[5] M.-J. Condorcet, *Éssai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*, Imprimiere Royale, Paris, 1785.

[6] D. Coppersmith, L. Fleischer, and A. Rudra, *Ordering by weighted number of wins gives a good ranking for weighted tournaments*, ACM Trans. Algorithms, 6 (2010), 55.

[7] P. Diaconis and R. Graham, *Spearman's footrule as a measure of disarray*, J. Roy. Statist. Soc. Ser. B, 39 (1977), pp. 262–268.

[8] C. Dwork, R. Kumar, M. Naor, and D. Sivakumar, *Rank aggregation methods for the web*, in Proceedings of the 10th International World Wide Web Conference, ACM, New York, 2001, pp. 613–622.

[9] R. Fagin, R. Kumar, M. Mahdian, D. Sivakumar, and E. Vee, *Comparing and aggregating rankings with ties*, in Proceedings of the 23rd Symposium on Principles of Database Systems, ACM, New York, 2004, pp. 47–58.

[10] L. Hemaspaandra, H. Spakowski, and J. Vogel, *The complexity of Kemeny elections*, Theoret. Comput. Sci., 349 (2005), pp. 382–391.

[11] J. Kemeny, *Mathematics without numbers*, Daedalus, 88 (1959), pp. 577–591.

[12] M. G. Kendall, *The treatment of ties in ranking problems*, Biometrika, 33 (1945), pp. 239–251.

[13] C. Kenyon-Mathieu and W. Schudy, *How to rank with few errors*, in Proceedings of the 39th Annual ACM Symposium on Theory of Computing, ACM, New York, 2007, pp. 95–103.

[14] F. Schalekamp and A. van Zuylen, *Rank aggregation: Together we're strong*, in Proceedings of the 11th Workshop on Algorithm Engineering and Experiments, SIAM, Philadelphia, 2009, pp. 38–51.

[15] C. Spearman, *Footrule for measuring correlation*, British J. Psychiatry, 2 (1906), pp. 89–108.

[16] A. van Zuylen and D. P. Williamson, *Deterministic pivoting algorithms for constrained ranking and clustering problems*, Math. Oper. Res., 34 (2009), pp. 594–620.

[17] H. P. Young and A. Levenglick, *A consistent extension of Condorcet's election principle*, SIAM J. Appl. Math., 35 (1978), pp. 285–300.