# Multi-Structural Games and Number of Quantifiers

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***Abstract*—We study multi-structural games, played on two sets $\mathcal{A}$ and $\mathcal{B}$ of structures. These games generalize Ehrenfeucht-Fraïssé games. Whereas Ehrenfeucht-Fraïssé games capture the quantifier rank of a first-order sentence, multi-structural games capture the number of quantifiers, in the sense that Spoiler wins the $r$-round game if and only if there is a first-order sentence $\phi$ with at most $r$ quantifiers, where every structure in $\mathcal{A}$ satisfies $\phi$ and no structure in $\mathcal{B}$ satisfies $\phi$. We use these games to give a complete characterization of the number of quantifiers required to distinguish linear orders of different sizes, and develop machinery for analyzing structures beyond linear orders.**

## 1. Introduction

Model theory has a number of techniques for proving inexpressibility results. However, as noted in [7], almost none of the key theorems and tools of model theory, such as the compactness theorem and the Löwenheim-Skolem theorems, apply to finite structures. Among the few tools of model theory that yield inexpressibility results for finite structures are Ehrenfeucht-Fraïssé games [5], [10], henceforth E-F games.

The standard E-F game is played by "Spoiler" and "Duplicator" on a pair $(A, B)$ of structures over the same first-order vocabulary $\tau$, for a specified number $r$ of *rounds*. In each round, Spoiler chooses an element from $A$ or from $B$, and Duplicator replies by choosing an element from the other structure. In this way, they determine sequences of elements $a_1, \ldots, a_r \in A$ and $b_1, \ldots, b_r \in B$, repetitions allowed, which in turn define substructures $A'$ of $A$ and $B'$ of $B$. Duplicator wins if the function $f(a_i) = b_i$ for $i = 1, \ldots, r$ is an isomorphism of $A'$ and $B'$. Otherwise, Spoiler wins.

The equivalence theorem for E-F games [5], [10] characterizes the minimum *quantifier rank* of a sentence $\phi$ over $\tau$ that is true for $A$ but false for $B$. The quantifier rank $qr(\phi)$ is defined as zero for a quantifier-free sentence $\phi$, and inductively:

$$\begin{aligned} qr(\neg\phi) &= qr(\phi), \\ qr(\phi \odot \psi) &= \max\{qr(\phi), qr(\psi)\}, \\ qr(Qx\phi) &= qr(\phi) + 1, \end{aligned}$$

where $\odot$ is any of the binary connectives and $Q$ is $\exists$ or $\forall$.

**Theorem 1.1.** ***Equivalence Theorem for E-F Games:*** *Spoiler wins the $r$-round E-F game on $(A, B)$ if and only if there is a sentence $\phi$ of quantifier rank at most $r$ such that $A \models \phi$ while $B \models \neg\phi$.*

The "if" direction of this theorem is fairly easy to prove by induction on $r$. This is the "useful" direction, which is used to prove inexpressibility results. The "only if" direction is somewhat tricky to prove; intuitively, it tells us that any technique for proving that a certain property cannot be defined by a first-order formula with a certain quantifier rank can, in principle, be replaced by a proof via E-F games. See [17], [18] for a proof and extended discussion.

We investigate a variant of E-F games that we call *multi-structural games*. These games make Duplicator more powerful and characterize the *number* of quantifiers rather than quantifier *rank*. It is straightforward to see that the minimum number of quantifiers needed to define a property $P$ is the same as the minimum size of the quantifier prefix of a sentence in prenex normal form that is needed to define property $P$. This is because converting a sentence into prenex normal form does not increase the number of quantifiers.

An equivalent of our multi-structural game was described in the journal version of Neil Immerman's paper "Number of Quantifiers Is Better Than Number of Tape Cells" [16]. Its conference version [15] did not mention the game. In [16], Immerman called it the "separability game" and showed that it characterized the number of quantifiers, without providing further results. We did not know Immerman had introduced these games when we first wrote our paper. Interestingly, in a first meeting with us, Immerman mentioned a related size game [1] but he did not remember introducing the separability game until we mentioned it in a second meeting with him [14].

Just prior to the conclusion of [16], Immerman remarked, "Little is known about how to play the separability game. We leave it here as a jumping off point for further research. We urge others to study it, hoping that the separability game may become a viable tool for ascertaining some of the lower bounds which are 'well believed' but have so far escaped proof."

Indeed, as our paper shows, analysis of the multi-structural games is often quite confounding, with many delicate issues.

We now define the rules of the multi-structural game. There are again two players, Spoiler and Duplicator, and there is a fixed number $r$ of rounds. Instead of being played on a pair $(A, B)$ of structures with the same vocabulary (as in an E-F game), it is played on a pair $(\mathcal{A}, \mathcal{B})$ of sets of structures, all with the same vocabulary. For $k$ with $0 \leq k \leq r$, by a

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*labeled structure* after $k$ rounds, we mean a structure along with a labeling of which elements were selected from it in each of the first $k$ rounds. Let $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{B}_0 = \mathcal{B}$. Thus, $\mathcal{A}_0$ represents the labeled structures from $\mathcal{A}$ after 0 rounds, and similarly for $\mathcal{B}_0$. If $1 \leq k < r$, let $\mathcal{A}_k$ be the labeled structures originating from $\mathcal{A}$ after $k$ rounds, and similarly for $\mathcal{B}_k$. In round $k+1$, Spoiler either chooses an element from each member of $\mathcal{A}_k$, thereby creating $\mathcal{A}_{k+1}$, or chooses an element from each member of $\mathcal{B}_k$, thereby creating $\mathcal{B}_{k+1}$. Duplicator responds as follows. Suppose that Spoiler chose an element from each member of $\mathcal{A}_k$, thereby creating $\mathcal{A}_{k+1}$. Duplicator can then make multiple copies of each labeled structure of $\mathcal{B}_k$, and choose an element from each copy, thereby creating $\mathcal{B}_{k+1}$. Similarly, if Spoiler chose an element from each member of $\mathcal{B}_k$, thereby creating $\mathcal{B}_{k+1}$, Duplicator can then make multiple copies of each labeled structure of $\mathcal{A}_k$, and choose an element from each copy, thereby creating $\mathcal{A}_{k+1}$. Duplicator wins if there is some labeled $A$ in $\mathcal{A}_r$ and some labeled $B$ in $\mathcal{B}_r$ where the labelings give a partial isomorphism. That is, if $a_i$ is the point selected in $A$ in round $i$, for $1 \leq i \leq r$, and if $b_i$ is the point selected in $B$ in round $i$, for $1 \leq i \leq r$, and if we let $A'$ be the substructure of $A$ (ignoring the labelings) generated by $\{a_1, \ldots, a_r\}$ and $B'$ the substructure of $B$ (ignoring the labelings) generated by $\{b_1, \ldots, b_r\}$, then the function $f$ where $f(a_i) = b_i$ for $i = 1, \ldots, r$ is an isomorphism of $A'$ and $B'$. Otherwise, Spoiler wins.

Note that on each of Duplicator's moves, Duplicator can make "every possible choice," via the multiple copies. Making every possible choice creates what we call the *oblivious strategy*. It is easy to see that Duplicator has a winning strategy if and only if the oblivious strategy is a winning strategy.

We shall prove the following theorem. It is analogous to Theorem 1.1 for ordinary E-F games.

**Theorem 1.2.** ***Equivalence Theorem for Multi-Structural Games:*** *Spoiler wins the $r$-round multi-structural game on $(\mathcal{A}, \mathcal{B})$ if and only if there is a sentence $\phi$ with at most $r$ quantifiers such that $A \models \phi$ for every $A \in \mathcal{A}$ while $B \models \neg\phi$ for every $B \in \mathcal{B}$.*

We now give an interesting refinement of the Equivalence Theorem (although, as we shall discuss, it does not seem to directly imply the Equivalence Theorem). Let $Q_1 \cdots Q_r$ be a sequence of quantifiers. We now define the "$Q_1 \cdots Q_r$ multi-structural game". It is an $r$-round multi-structural game, with the following restrictions on Spoiler. If $Q_k$ is an existential quantifier, then Spoiler's $k$th move must be in $\mathcal{A}$, and otherwise must be in $\mathcal{B}$. We then have the following result.

**Theorem 1.3.** ***Fixed Prefix Equivalence Theorem for Multi-Structural Games:*** *Spoiler wins the $Q_1 \cdots Q_r$ multi-structural game on $(\mathcal{A}, \mathcal{B})$ if and only if there is a sentence $\phi$ in prenex normal form with exactly r quantifiers, in the order $Q_1 \cdots Q_r$, such that $A \models \phi$ for every $A \in \mathcal{A}$ while $B \models \neg\phi$ for every $B \in \mathcal{B}$.*

On the face of it, Theorem 1.3 does not seem to directly imply Theorem 1.2, for the following reason. It is a priori conceivable (if we did not know the Equivalence Theorem) that Spoiler's winning strategy in an $r$-round game is to move first in $\mathcal{A}$, and then, depending on Duplicator's response, to move in either $\mathcal{A}$ or $\mathcal{B}$. So there is then no prefix $Q_1 \cdots Q_r$ dictating where Spoiler must move. In fact, in an E-F game (but not in a multi-structural game), it can indeed happen that Spoiler can win in 2 rounds, and where Spoiler plays in the second round depends on how Duplicator played in the first round. An example of where this phenomenon can take place in a 2-round E-F game is via the sentence $\exists x(\forall y B(x,y) \wedge \exists y R(x,y))$.

The proof of Theorem 1.2 appears in Section 2. The proof of Theorem 1.3 is almost the same as the proof of Theorem 1.2, with very minor changes.

There is an interesting and non-obvious difference between E-F games and multi-structural games. Let us say that a player makes a move "on top of" a previous move if the player selects an element $c$ of a structure, and the same element $c$ had been selected by either player in an earlier round. It is easy to see that in an E-F game, it never helps Spoiler to make a move on top of a previous move (it only wastes a round). On the other hand, in multi-structural games the issue of playing on top of a previous move is a frequent consideration for us. In fact, a detailed analysis shows that playing a move on top of a previous move may be a necessary part of a winning Spoiler strategy. See the full version of this paper [8, Appendix D].

We now give an example (Figure 1) that shows differences between the E-F game and the multi-structural game, and what they say about quantifier rank vs. number of quantifiers. Consider the following two structures $B$ (for "Big") and $L$ (for "Little"), over $\tau = \{<\}$, where $<$ is the binary "less than" relation. The vertex labels are not part of the structures. Elements that appear to the left are considered to be less than elements to the right. $B$ is a linear order on 3 elements and $L$ is a linear order on two elements. In the text of this paper we write $B(i)$ (or $L(i)$) to denote the $i$th element in the linear order $B$ (respectively, $L$), while in the figures, for economy of space, we label the $i$th vertex instead by $Bi$ (respectively, $Li$). Further, rather than use the notation $<(x,y)$ we shall use


**Fig. 1.** An example showing the difference between multi-structural and E-F games.

the customary $x < y$ notation.

Suppose first that the number $r$ of rounds is 2. We show that Spoiler wins the E-F game. On Spoiler's first move, Spoiler selects vertex $B(2)$ in $B$. Duplicator must select either $L(1)$ or $L(2)$ in $L$. If Duplicator chooses $L(1)$, then Spoiler selects $B(1)$ in $B$. After Duplicator selects $L(2)$ in $L$ (her only legal move), Spoiler wins, since the mapping given by $B(2) \mapsto L(1)$ and $B(1) \mapsto L(2)$ fails to be a partial isomorphism because the "less than" relationship is flipped. If Duplicator had instead selected $L(2)$ in the first round, then Spoiler would have won, by a similar argument, by selecting $B(3)$ in $B$ in the second round.

The fact that Spoiler wins the 2-round game over $(B, L)$ tells us (by Theorem 1.1) that there is a sentence $\phi$ of quantifier rank at most 2 such that $B \models \phi$ while $L \models \neg\phi$. Such a sentence is: $\exists x(\exists y(y < x) \ \wedge \ \exists y(x < y))$.

Now let us consider the 2-round multi-structural game over $(\mathcal{B}, \mathcal{L})$, where $\mathcal{B} = \{B\}$ and $\mathcal{L} = \{L\}$. We show that unlike the 2-round E-F game, in the 2-round multi-structural game, Duplicator wins. It is easy to see that Duplicator wins if Spoiler's first-round move is anything other than $B(2)$. Let us see what happens if Spoiler's first move is $B(2)$, which was a winning move for Spoiler in the 2-round E-F game. Then on Duplicator's first move, Duplicator makes a second copy of $L$ and in one copy, call it $L_1$, Duplicator selects $L(1)$, and in the other copy, call it $L_2$, Duplicator selects $L(2)$. Let us now consider Spoiler's possible second round responses. Suppose first that Spoiler's second round move is in $B$. If Spoiler selects $B(3)$, then Duplicator selects $L(2)$ in $L_1$ and the mapping $B \rightarrow L_1$ such that $B(2) \mapsto L(1), B(3) \mapsto L(2)$ yields a partial isomorphism. On the other hand, if Spoiler selects $B(1)$, then Duplicator selects $L(1)$ in $L_2$ and $B \rightarrow L_2$ such that $B(1) \mapsto L(1), B(2) \mapsto L(2)$ yields a partial isomorphism. Section 4 will complete the analysis of a Duplicator win. Since Duplicator wins the 2-round game, it follows by Theorem 1.2 there is no formula with just two quantifiers that distinguishes $\mathcal{B}$ from $\mathcal{L}$.

The focus of our analysis of multi-structural games in this paper is to linear orders. In the case of E-F games, one has the following:

**Theorem 1.4.** *Let $f(r) = 2^r - 1$. In an $r$-round E-F game played on two linear orders of different sizes, Duplicator wins if and only if the size of the smaller linear order is at least $f(r)$.*

Since part of the proof of Theorem 1.4 is left to an exercise in [22], we give a proof in [8, Appendix A]. Further, the proof illustrates a simple recursive idea that is surprisingly not available to us in the analysis of linear orders from the vantage point of multi-structural games.

Theorem 1.4 together with Theorem 1.1 imply that $f(r)$ is the maximum value $k$ such that a sentence of quantifier rank $r$ can distinguish linear orders of size $k$ and above from those of size smaller than $k$.

In analogy to this function $f$, and in an effort to arrive at a parallel theorem to Theorem 1.4, we make the following definition.

**Definition 1.5.** *Define the function $g : \mathbb{N} \rightarrow \mathbb{N}$ such that $g(r)$ is the maximum number $k$ such that there is a formula with $r$ quantifiers that can distinguish linear orders of size $k$ or larger, from linear orders of size less than $k$.*

To see that $g$ is well defined, observe that the sentence

$$\exists x_1 \cdots \exists x_r \bigwedge_{1 \leq i < r} x_i < x_{i+1}, \tag{1}$$

distinguishes linear orders of size $r$ or larger from linear orders of size less than $r$. Furthermore, there are only finitely many inequivalent sentences with up to $r$ quantifiers that include only the relation symbols $<$ and $=$, some fraction of which distinguish linear orders of some size $k$ or greater from linear orders of size less than $k$. There is therefore a maximum such $k \geq r$, which is then $g(r)$.

After building up quite a bit of machinery we eventually arrive at the following:

**Theorem 1.6.** *The function $g$ takes on the following values: $g(1) = 1, g(2) = 2, g(3) = 4, g(4) = 10$, and for $r > 4$,*

$$g(r) = \begin{cases} 2g(r-1) & \textit{if } r \textit{ is even,} \\ 2g(r-1)+1 & \textit{if } r \textit{ is odd.} \end{cases}$$

The value $g(3) = 4$ is a curious anomaly. If it had turned out that $g(3) = 5$, then the entire induction could be founded on $r = 1$. The proof of Theorem 1.6 is a careful mathematical journey to restore this induction founded instead at $r = 4$.

The following theorem for multi-structural games is the analog of Theorem 1.4 for E-F games, and describes precisely when Duplicator (alternatively, Spoiler) wins $r$-round multi-structural games on two linear orders of different sizes.

**Theorem 1.7.** *In an $r$-round multi-structural game played on two linear orders of different sizes Duplicator has a winning strategy if and only if the size of the smaller linear order is at least $g(r)$.*

It is important to note that neither the *if* nor *only if* portion of this theorem is implied by the definition of $g$.

Both E-F games and multi-structural games are used to prove inexpressibility results by showing there is a wining strategy for Duplicator. It is typically easier to demonstrate a winning strategy for Duplicator in E-F games than in multi-structural games, for several reasons. First, it is easier to reason about only two structures at a time rather than about many structures at a time. Second, in multi-structural games, there is a tactic available to Spoiler (that is of no use in E-F games) to make one move on top of an earlier move in one of the structures, and this can greatly complicate the analysis. On the other hand, in mutii-structural games, Duplicator has the advantage of being able to make multiple copies of structures and make different moves on the various copies. This feature can be very useful in proving a winning strategy for Duplicator.

A similar phenomenon of modifying the rules of the game to make it easier for Duplicator to win arose in defining and making use of Ajtai-Fagin games [2] rather than making use of the originally defined Fagin games [6] for proving inexpressibility results in monadic existential second-order logic (called "monadic NP" in [9]). In Fagin games, there is a coloring round (a choice of the combinations of existentially-quantified monadic predicates) where Spoiler colors $A$ then Duplicator colors $B$, and then an ordinary E-F game is played on the colored structures. In Ajtai-Fagin games, Spoiler must commit to a coloring of $A$ without knowing what the other structure $B$ is. Fagin, Stockmeyer, and Vardi [9] use Ajtai-Fagin games to give a much simpler proof that connectivity is not in monadic NP than Fagin's original proof in [6]. In extending multi-structural games to second-order logic, which we think is an interesting and important future step (and which is straightforward to define), these games can easily simulate Ajtai-Fagin games. This is because we can replace structure $A$ by the singleton set $\mathcal{A} = \{A\}$, and replace the structure $B$ by a set $\mathcal{B}$ that contains all possible choices for $B$ that Duplicator might choose in the Ajtai-Fagin game after Spoiler colors $A$.

### *A. Related work*

Since our results can be viewed as giving information about the size of prefixes of sentences in prenex normal form, we begin by discussing some other papers that focus on such prefixes.

Rosen [21] shows that there is a strict prefix hierarchy, based on the prefixes of sentences written in prenex normal form. The proof involves standard E-F games.

Dawar and Sankaran [4] consider E-F games, each of which focuses on a fixed prenex prefix. For example, there is one game that deals with the prenex prefix $\exists\forall\exists$. For each of these prefixes, they define an E-F game on a pair $(A, B)$ of structures. For example, in the $\exists\forall\exists$ game, Spoiler must move first in $A$, then in $B$, and then in $A$. Their Theorem 2.3 says that Spoiler has a winning strategy in a prefix game if and only if there is a sentence in prenex normal form with exactly that prefix that is true about $A$ but not about $B$. Unfortunately, the "only if" direction of their Theorem 2.3 is false [3]. This is because if $A$ is a linear order of size 5 and $B$ is a linear order of size 4, and the prefix is $\exists\forall\exists$, then it turns out that Spoiler wins that 3-round prefix game, but it follows from our Theorem 1.6 that $A$ and $B$ agree on all sentences with at most three quantifiers, and in particular on all sentences $\exists x\forall y\exists z\phi(x, y, z)$, where $\phi$ is quantifier-free. Fortunately, in their paper, Dawar and Sankaran just make use of the "if" direction of Theorem 2.3, which is correct [3].

We now discuss some papers that, like ours, modify E-F games by allowing a pair of sets of structures, rather than simply a pair of structures. Adler and Immerman [1] use a type of E-F game that involve a pair of sets of structures, where, as in our multi-structural games, Duplicator can make multiple copies of structures and make different moves on them. Adler and Immerman's concern is to obtain results about the size of sentences (rather than the number of quantifiers) in transitive closure logic (first-order logic with the transitive closure operator). The rules of the game are rather complicated, since it must deal with transitive closure logic and capture the size of sentences.

Hella and Vilander [13] build on Adler and Immerman's game, and their goal is also to determine sentence size (but in modal logic). Their rules of their game are also fairly complicated.

Grohe and Schweikardt [11] introduce a method (extended syntax trees) that corresponds to a game tree that is constructed by the two players in the Adler-Immerman game. They use these to study the size of formulas in the $2, 3$ and $4$-variable fragments of first-order logic on linear orders.

Lotfallah [19] has a class of E-F games played on a pair of sets of structures rather than on a pair of single structures. In Lotfallah's games, Duplicator cannot make multiple copies of structures. A follow-up paper by Lotfallah and Youssef [20] characterizes certain first and second order prefix types but does not involve sets of structures.

Hella and Väänänen [12], like Lotfallah, have a class of E-F games played on a pair of sets of structures rather than on a pair of single structures, where Duplicator cannot make multiple copies of structures. Hella and Väänänen characterize the size of formulas needed for separating sets of structures in propositional logic and a second one for first-order logic. The first-order game is used for proving an exact bound on the size of existential formulas needed to define the length of linear orders.

### *B. Overview of the Sections*

In Section 2 we prove Theorem 1.2. In Section 3 we establish lower bounds for the values of $g(r)$ associated with Theorem 1.6. In Section 4 we start by establishing upper bounds on $g(r)$, for $r = 1, 2$ and 3. The natural next step would be to proceed to higher values of $r$ using a type of recursive argument, but in Section 5 we show why the natural recursive argument for multi-structural games (henceforth "MS games") does not work. In Section 6 we introduce a new type of game (an MS game with "atoms") which allows us to recurse and prove upper bounds for all $r$. While games with "atoms" are an important part of our proof, the final sentences guaranteed by Definition 1.5 and Theorem 1.6 do not contain atoms. The upper bounds are then seen to be tight with respect to our lower bounds and hence, in Section 7, we are able to prove Theorem 1.6. The machinery that we have built up then enables a quick proof of Theorem 7.2, which is just the syntactic equivalent of Theorem 1.7 from the Introduction (there stated game theoretically). We present our conclusions in Section 8.

## 2. Proof of the Equivalence Theorem

To prove Theorem 1.2, we treat the labeling of elements by assigning special constants $c_1, \ldots, c_r$ that are added to the vocabulary $\tau$ in order to maintain the induction. Note that $\tau$ may have its own constants. We write $(A; \ldots; c_1 \leftarrow a_1)$ to mean the structure obtained after assigning $c_1$ to the element $a_1$ in $A$, and so on. For each $i$ with $0 \leq i \leq r$, we write $A \sim_i B$ to mean that after $i$ rounds of the game, the substructures obtained by restricting to the domain $\{a_1, \ldots, a_i\}$ of points, corresponding to the constants $c_1, \ldots, c_i$, are isomorphic. Then $A \sim_r B$ has the same effect as defining the function $f(a_i) = b_i$, giving the partial isomorphism as previously stated. We continue writing $A, B, \ldots$ for both structures and their underlying sets.

*Proof of Theorem 1.2.* Both directions are proved by induction on the number $r$ of rounds. For $r = 0$, Spoiler winning in $0$ rounds means that for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the restrictions $A', B'$ of $A$ and $B$ (respectively) to their constants must be non-isomorphic. For every $A \in \mathcal{A}$, we can write a quantifier-free sentence $\phi_A$ that characterizes $A'$ up to isomorphism, using equality to identify any coinciding constants. Then take $\phi$ to be the disjunction of all $\phi_A$ – note that even if $\mathcal{A}$ is infinite, there are only finitely many distinct $\phi_A$. Then $A \models \phi$ for all $A \in \mathcal{A}$. Now consider any $B \in \mathcal{B}$. We claim that $B \models \neg\phi$. As $\neg\phi$ is a conjunction, this is equivalent to $B \models \neg\phi_A$ for every $A$. Suppose not, then we would have $B \not\models \neg\phi_A$, i.e., $B \models \phi_A$. But because $\phi_A$ characterizes $A'$ up to isomorphism, this would make $B'$ isomorphic to $A'$, contradicting that Spoiler wins. Hence $\phi$ is a quantifier-free sentence that distinguishes $\mathcal{A}$ and $\mathcal{B}$.

Conversely, if there is a quantifier-free sentence $\phi$ that distinguishes $\mathcal{A}$ and $\mathcal{B}$, then there cannot exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that the restrictions $A'$ and $B'$ to the constants appearing in $\phi$ are isomorphic. Thus, Duplicator has already lost at the outset of the game played on the pair $(\mathcal{A}, \mathcal{B})$, without any of the special labeling constants having been introduced.

Now suppose $r \geq 1$ and the equivalence is true for $r-1$. We induct on the first move rather than the last move of the games. For the forward direction, suppose Spoiler can win—say by playing in $\mathcal{B}$. For each $B \in \mathcal{B}$, Spoiler selects an element $b \in B$ and assigns the special constant $c_1$ to it. Duplicator replies by replicating every $A \in \mathcal{A}$ over all possible choices of $c_1 \in A$. The resulting game position $(\mathcal{A}^1, \mathcal{B}^1)$ is winnable in $r-1$ rounds for Spoiler. By induction hypothesis, there is a sentence $\psi$ with $r-1$ quantifiers that distinguishes $\mathcal{B}^1$ from $\mathcal{A}^1$. Now define $\phi = (\exists x_1)\psi'$ where $\psi'$ replaces all occurrences of $c_1$ in $\psi$ by $x_1$. (This is all right even in degenerate cases where $c_1$ does not occur in $\psi$.) For every $B \in \mathcal{B}$, we have $B \models \phi$ because there is a $b \in B$ such that $(B; \ldots; c_1 \leftarrow b) \models \psi$, namely the $b$ that Spoiler played in $B$. Hence it suffices to show that $A \models \neg\phi = (\forall x_1)\neg\psi'$ for every $A \in \mathcal{A}$. After Duplicator's play, $A$ was replaced by

$$\{(A; \ldots; c_1 \leftarrow a_1), (A; \ldots; c_1 \leftarrow a_2), \ldots, (A; \ldots; c_1 \leftarrow a_m)\},$$

where $A = \{a_1, \ldots, a_m\}$. Since $\psi$ distinguishes $\mathcal{B}^1$ from $\mathcal{A}^1$, we have $(A; \ldots, c_1 \leftarrow a_j) \models \neg\psi$ for each $j = 1, \ldots, m$. It follows that $A \models (\forall x_1)\neg\psi'$. The case where Spoiler wins by playing in $\mathcal{A}$ is handled symmetrically.

Going the other way, suppose $\phi$ is a prenex formula with $r$ quantifiers that distinguishes $\mathcal{A}$ from $\mathcal{B}$. If the leading quantifier is $\forall$ then $\neg\phi$ has leading quantifier $\exists$ and distinguishes $\mathcal{B}$ from $\mathcal{A}$, so we can reason by symmetry. So let $\phi = (\exists x_1)\psi'$ for some $\psi'$ and take $\psi$ to be the sentence with $x_1$ replaced everywhere by the special constant symbol $c_1$. For every $A \in \mathcal{A}$, $A \models \phi$, so there exists $a_1 \in A$ such that $(A; \ldots; c_1 \leftarrow a_1) \models \psi$. Spoiler can play such an element $a_1$ in every $A$. Now every $B \in \mathcal{B}$ models $\neg\phi = (\forall x_1)\neg\psi'$. For every $b \in B$, Duplicator creates the structure $(B; \ldots; c_1 \leftarrow b)$, but regardless of $b$, it models $\neg\psi$. Thus, $\psi$ distinguishes the resulting set $\mathcal{A}^1$ from Duplicator's $\mathcal{B}^1$, has $r-1$ quantifiers, and includes $c_1$ along with any previous constants. By induction hypothesis, Spoiler wins from $(\mathcal{A}^1, \mathcal{B}^1)$ in $r-1$ rounds, so Spoiler wins from $(\mathcal{A}, \mathcal{B})$ in $r$ rounds. □

**Definition 2.1.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two sets of structures. Write $\mathcal{A} \equiv_r \mathcal{B}$ iff Duplicator has a winning strategy for MS games of $r$ rounds on $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{A} = \{A\}$ and $\mathcal{B} = \{B\}$ we also write $A \equiv_r B$.*

An important consequence of Theorem 1.2 is the following.

**Lemma 2.2.** *The relation $\equiv_r$ is an equivalence relation between sets of structures.*

*Proof.* That the relation $\equiv_r$ is reflexive and symmetric follows immediately from the definition. For transitivity, suppose there are three sets of structures, $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ such that $\mathcal{A} \equiv_r \mathcal{B}$ and $\mathcal{B} \equiv_r \mathcal{C}$. By the Equivalence Theorem 1.2, $\mathcal{A} \equiv_r \mathcal{B}$ implies that $\mathcal{A}$ and $\mathcal{B}$ agree on the same sentences with at most $r$ quantifiers. Similarly, $\mathcal{B} \equiv_r \mathcal{C}$ implies that $\mathcal{B}$ and $\mathcal{C}$ agree on the same set of sentences with at most $r$ quantifiers. Hence, $\mathcal{A}$ and $\mathcal{C}$ agree on these same set of sentences. By the Equivalence Theorem again, it follows that $\mathcal{A} \equiv_r \mathcal{C}$. □

## 3. Lower Bounds on $g(r)$

As is standard in model theory, we assume a non-empty universe, and so all linear orders are of size at least 1.

**Lemma 3.1.** $g(1) = 1$.

*Proof.* The sentence $\exists x(x = x)$ is true for all linear orders so that $g(1) \geq 1$. Moreover, Duplicator can win 1-round MS games whenever both linear orders are of size 1 or greater, which implies that $g(1) \leq 1$. □

**Lemma 3.2.** $g(2) \geq 2$

*Proof.* The sentence $\Phi_2 = \exists x \exists y(x < y)$ distinguishes linear orders of size 2 and above, from the linear order of size 1. □

**Lemma 3.3.** $g(3) \geq 4$

*Proof.* The following sentence, with 3 quantifiers, distinguishes linear orders of size at least 4 from those of size at most 3:

$$\Phi_3 = \forall x \exists y \exists z(x < y < z \vee y < z < x) \tag{2}$$

□

**Lemma 3.4.** $g(4) \geq 10$.

*Proof.* The following sentence with 4 quantifiers distinguishes linear orders of size at least 10 from those of size at most 9.

$$\begin{aligned}
\Phi_4 = \forall x \exists y \forall z \exists w (& \\
& x < z < y \rightarrow (w \neq z \wedge x < w < y) \quad \wedge & (3)\\
& x < y < z \rightarrow (w \neq z \wedge x < y < w) \quad \wedge & (4)\\
& y < z < x \rightarrow (w \neq z \wedge y < w < x) \quad \wedge & (5)\\
& z < y < x \rightarrow (w \neq z \wedge w < y < x) \quad \wedge & (6)\\
& z = x \rightarrow (x < w < y \vee y < w < x) \quad \wedge & (7)\\
& z = y \rightarrow (x < y < w \vee w < y < x)). & (8)
\end{aligned}$$

This sentence captures the fact that "for every $x$ there is a $y$ with two or more elements on each side of $y$, both of which are on the same side of $x$ as $y$", a fact that is true for linear orders of size 10 or greater, but not for linear orders of size less than 10. For example, in a linear order of size 9, the middle element will not have an element to either side of it having these properties. More specifically, the first four implications (3)–(6) say that for every $x$ there is a $y$ such that for any $z \neq x, y$, with $z$ on the same side of $x$ as $y$, there is an additional element besides $z$ on the same side of $y$ and also on the same side of $x$ as $y$. The last two implications (7)–(8) are critical and insure that there are elements (i) between $x$ and $y$ and (ii) less than $y$ if $y < x$, and greater than $y$ if $y > x$. □

For a game-based proof of Lemma 3.4 see [8, Appendix D]. Before proceeding further let us establish some terminology that is intended to make the reading smoother. In cases where there are multiple linear orders on one side or another we often refer to the different linear orders as different "boards" that Spoiler and Duplicator play on.

When we say a "$K$ versus $K'$ game" or a "$K$ vs. $K'$ game", we mean an MS game played on $(\mathcal{A}, \mathcal{B})$ where $\mathcal{A}$ consists of a single linear order of size $K$, and $\mathcal{B}$ consists of a single linear order of size $K'$.

We will typically play games where $\mathcal{A}$ and $\mathcal{B}$ each consist of a single linear order as above. In this context, as we did in the Introduction, we will use $B$ to denote the *big* linear order and $L$ to denote the *little* linear order.

With initial values $g(1) = 1, g(2) \geq 2$ (Lemma 3.2), $g(3) \geq 4$ (Lemma 3.3) and $g(4) \geq 10$ (Lemma 3.4), we establish all remaining lower bounds via a game argument:

**Theorem 3.5.** *For $r > 4$,*

$$g(r) \geq \begin{cases} 2g(r-1) & \text{if } r \text{ is even,} \\ 2g(r-1)+1 & \text{if } r \text{ is odd.} \end{cases} \tag{9}$$

*Proof.* Let us establish (9) first for odd $r$. To establish the theorem (for odd $r$) we need to provide a Spoiler-winning strategy for a game with arbitrary linear orders of sizes at least $2g(r-1)+1$ on one side and arbitrary linear orders of sizes at most $2g(r-1)$ on the other side. For simplicity, we will show the strategy for the game in which we have a single linear order of size at least $2g(r-1)+1$ on one side and a single linear order of size at most $2g(r-1)$ on the other side. The strategy for the case of multiple linear orders on each side is exactly the same, as we shall see. Analogous remarks hold for the even $r$ case.

Suppose we have linear orders of size $2g(r-1)+1$ and $2g(r-1)$ as in Figure 2. We describe a winning strategy for Spoiler. Spoiler begins by playing the middle element on $B$, i.e. $B(g(r-1)+1)$. Duplicator responds playing every possible move leaving short sides that all have fewer than $g(r-1)$ elements. Spoiler next makes his 2nd round move on each board of $L$ on the short sides, while playing on top of any end moves (moves without clearly defined short sides), such as those shown on the bottom two boards in Figure

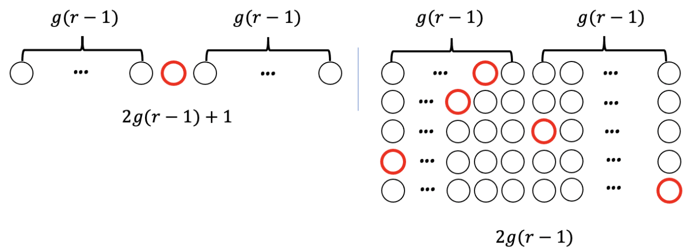


**Fig. 2.** The odd $r$ case: linear orders of sizes $2g(r-1)+1$ and $2g(r-1)$. Spoiler plays first on $B$ (the left hand side). First round moves are indicated in red. Duplicator responds playing every possible move. Five exemplary boards with different moves played on each are shown.

2. With the exception of the two end moves, which we will handle independently, Spoiler is going to play all subsequent moves entirely on the short sides of the boards of $L$ and the corresponding sides of the boards of $B$ – in other words if the short side of $L$ is on the left, he will play on the left side of $B$, and vice versa – in accordance with the known Spoiler-winning strategy on boards of size $g(r-1)$ or greater vs. boards of size less than $g(r-1)$.

In order to see how he is able to do this, we need to use a strong inductive assumption, namely that in games of $r$ rounds, when $r$ is even, and there are linear orders of sizes $g(r)$ or greater on one side, and less than $g(r)$ on the other side, Spoiler can then always win by playing first on the $L$ side. The game-based proof of Lemma 3.4 demonstrated such a strategy for the case $r = 4$ and we will have to keep this commitment when we cover additional even $r$ cases next. For now though, we are assuming that $r$ is odd, so $r - 1$ is even and we have such a strategy.

Duplicator will respond with the oblivious strategy, making many copies of $B$ and playing all possible moves. From this point forward the boards on both sides that have had their 2nd moves played to the right of the 1st moves conceptually constitute one game, and the boards on both sides that have had their 2nd moves played to the left of the 1st moves conceptually constitute a second game. The conceptual game played entirely on the left side of the boards can be won by Spoiler as well as the conceptual game played on the right (both times using the strong induction hypotheses). Note that since both games have the same larger size boards, they pick the same sides to play on in all rounds and hence can be played round-by-round in tandem. In so doing, all partial isomorphisms are broken and so Spoiler wins the combined game.

However, we have not yet described how to take care of the case where Duplicator played end moves on $L$ in the 1st round and Spoiler reciprocated by playing on top of these end moves. In order to maintain an isomorphism with either of these boards Duplicator will have to play on top of the 1st move on $B$ – which will break any potential isomorphism with any other $L$ boards. Now we use a key observation from the Spoiler-winning strategy in the 10+ vs. 9- game that established $g(4) \geq 10$ (see the proof of Lemma 3.4): in the next-to-last round Spoiler played on $L$, and in the last round he played on $B$. Note that the $r$-round Spoiler strategy recursively uses an $(r-1)$-round Spoiler strategy,..., eventually using the 4-round Spoiler strategy since the base case of this lemma is $r = 5$, which is defined in terms of $r = 4$. Thus, for boards that remain isomorphic long enough, Spoiler will play the last two rounds consecutively on $L$ and then $B$.

In the next-to-last round in which Spoiler plays on $L$, Spoiler will select any element to the right of $L(1)$ on the boards where $L(1)$ was played as a first move, and Spoiler will select any element to the left of $L(2g(r-1))$ on the boards where $L(2g(r-1))$ was played as a 1st move. As a result, to maintain partial isomorphisms with the boards in which, respectively, $L(1)$ and $L(2g(r-1))$ were played, Duplicator will have to play so that the $B$ boards that remain isomorphic with the board in which $L(1)$ was played are not isomorphic to the boards in which $L(2g(r-1))$ was played, and vice versa. Thus, in the final round, Spoiler will play to the right of the middle element on the $B$ boards that maintained a partial isomorphism with the boards where $L(2g(r-1))$ was played first, thus killing all surviving partial isomorphisms, and will play to the left of the middle element on the $B$ boards that maintained a partial isomorphism with boards where $L(1)$ was played first, killing all of its remaining partial isomorphisms.

For the more general case where $|B| > 2g(r-1)+1$, Spoiler simply picks any element having at least $g(r-1)$ elements on each side of his 1st move and the play proceeds via the same induction. Analogously, if $|L| < 2g(r-1)$ it still follows that Duplicator's 1st round moves leave a short side of size less than $g(r-1)$, and hence the argument remains the same with respect to such smaller size $L$.

Next let us tackle the case where $r$ is even. We begin as usual with the base case of boards of sizes $2g(r-1)$ and $2g(r-1)-1$. See Figure 3.

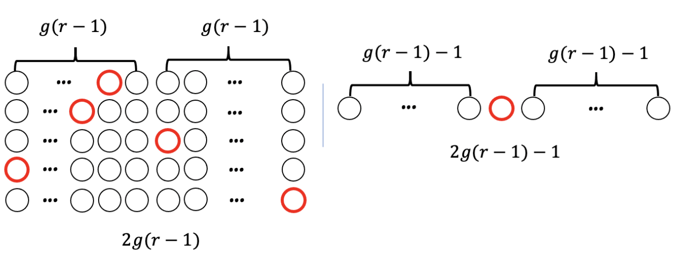


**Fig. 3.** The even $r$ case: linear orders of sizes $2g(r-1)$ and $2g(r-1)-1$. Spoiler plays first on $L$ (the right hand side). First round moves are indicated in red. Duplicator responds playing every possible move. Five exemplary boards with different moves played on each are shown.

As Spoiler, we must keep our inductive commitment to play first on $L$ and so play the middle element, as indicated in the figure. Now any move by Duplicator on $B$ leaves a long side of size at least $g(r-1)$ versus the same side (left or right of the 1st move) on $L$, of size $g(r-1)-1$. Thus we adapt the argument from the odd $r$ case to this case, but where we play on the long side now rather than on the short side. The strong induction hypothesis we need this time is that smaller odd $r$ cases can be won by playing the 1st round from $B$ – but we know this to be the case for $r = 5$, the very first case covered by this theorem, and all subsequent cases of odd $r$ by how we argued the odd $r$ case. The theorem therefore follows. □

For $r > 4$, the sentences associated with the strategies described in the proof of the above theorem say, respectively, for $r$ even, that for every $x$ there is a linear order of size at least $g(r-1)$ either to the left or right of $x$, and for $r$ odd, that there is an element $x$ with a linear order of size at least $g(r-1)$ both to the left and right of $x$.

## 4. Towards Establishing Upper Bounds on $g(r)$

A potent tool for finding an upper bound k on the value of $g(r)$ will be to find strategies such that Duplicator can win $r$-round games on two linear orders whenever the sizes of the linear orders are at least $k$. All of our upper bounds are established in this manner.

Since we have established that $g(1) = 1$, we start establishing upper bounds at $g(2)$.

**Lemma 4.1.** *Duplicator can win* 2*-round MS games whenever both linear orders are of size* 2 *or greater, and hence* $g(2) \leq 2$*.*

*Proof.* In the Introduction we considered a 2-round MS game on two linear orders of sizes $|B| = 3, |L| = 2$. Figure 1 is given again here for ease of reference.

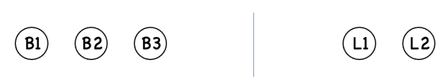


**Fig. 4.** The case $|B| = 3, |L| = 2$.

The Introduction covered the case where Spoiler selects the middle $B$ element, $B(2)$, in the first round. The case where Spoiler picks an end element from either the $L$ or $B$ boards in the first round is easier – Duplicator just picks the corresponding end element from the other linear order and she does not even need to make a second copy of the board to win. For example, in response to $B(1)$, Duplicator will play $L(1)$, or in response to $L(2)$, Duplicator will play $B(3)$, in either case leading to simple wins.

In the example given in the Introduction, where Spoiler played $B(2)$, once Duplicator makes copies and plays different moves in each copy, we render the game after round 1, as in Figure 5.


**Fig. 5.** In response to Spoiler playing $B(2)$, Duplicator makes a second copy of the $L$ board and place $L(1)$ on one board and $L(2)$ on the other board. (The diagrams omit parentheses.)

To complete the analysis, we must show that Duplicator wins whenever $2 \leq |L| < |B|$. Let us begin with the case

$|L| = 2$ and $|B| > 3$. If Spoiler picks an end element from $B$ or from $L$, then Duplicator picks the corresponding end element from the opposite side and wins. On the other hand, if Spoiler picks a non-end element on $B$, Duplicator picks $L(1)$ on top and $L(2)$ on bottom, winning just like in the introduction. We are left to consider the case when $3 \leq |L| < |B|$. Playing end moves from either $B$ or $L$ have the same effect as before. While if Spoiler picks a non-end element from either $B$ (or $L$), Duplicator wins by playing *any* non-end element from, respectively, $L$ (or $B$), guaranteeing a 2-round win. □

**Lemma 4.2.** *Duplicator can win 3-round MS games on linear orders whenever both linear orders are of size 4 or greater, and hence $g(3) \leq 4$.*

*Proof.* Let us start with the base case $|B| = 5, |L| = 4$. See Figure 6. Duplicator-winning outcomes associated with all

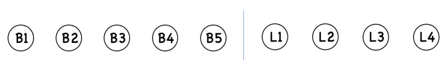


**Fig. 6.** The case $|B| = 5, |L| = 4$.

Spoiler 1st round plays, other than $B(3)$, are easy to analyze and are described in [8, Appendix B].

In response to the one tricky Spoiler 1st round move of $B(3)$, Duplicator creates additional copies of the $L$ board and makes every possible move, as depicted in Figure 7. Let's now analyze the possible Spoiler 2nd round responses.

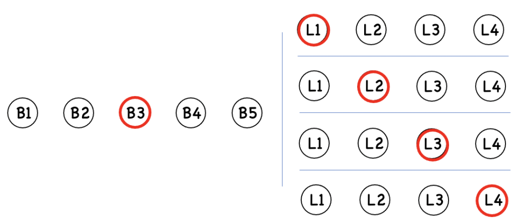


**Fig. 7.** After Spoiler plays $B(3)$ from the $B$ side, Duplicator plays $L(1)$, $L(2)$, $L(3)$ and $L(4)$ on different boards on the $L$ side.

Case where Spoiler makes 2nd round move on $B$:

| Round 2 | |
|---|---|
| **Spoiler** | **Duplicator** |
| $B(1)$ | $L(1)$ on 3rd $L$ board insuring a win on that board |
| $B(2)$ | $L(2)$ on 3rd $L$ board insuring a win on that board |
| $B(3)$ | $L(3)$ on 3rd $L$ board insuring a win on that board |
| $B(4)$ | $L(3)$ on 2nd $L$ board insuring a win on that board |
| $B(5)$ | $L(4)$ on 2nd $L$ board insuring a win on that board |

Case where Spoiler makes 2nd round moves on $L$:

First consider the possibility of Spoiler playing atop an existing move. If he plays atop $L(2)$ on board 2, or $L(3)$ on board 3, then Duplicator will play atop $B(3)$, ensuring victory on the associated boards in another move. Analogously, if Spoiler plays on top of *both* $L(1)$ on the top board and $L(4)$ on the bottom board, then Duplicator can play atop $B(3)$ guaranteeing a win on one of these $L$ boards or the other. Thus, we may assume Spoiler plays atop *at most* one of the existing moves, with that one move being atop either $L(1)$ or $L(4)$. Of the at least three boards in which Spoiler does *not* play on top of existing moves, the moves are either to the right of the existing moves, or to the left of the existing moves, and hence, there must be at least two played to one side or the other of the existing moves. Without loss of generality, assume that at least two of these moves are to the right of the existing moves.

Suppose that one of the moves to the right is on the 2nd board. An $L(3)$ move would be met by a $B(4)$ response by Duplicator, while an $L(4)$ move would be met by a $B(5)$ response, in either case leading to a single board win for Duplicator. Thus we can assume that the two moves to the right of the existing moves are on the 1st and 3rd boards – the only two boards we need to consider to conclude this bit of the analysis. The element $L(4)$ must be selected from board 3. If $L(2)$ is selected from board 1 then Duplicator wins by selecting $B(4)$: any 3rd round Spoiler move on $B$ is parried on one of the two $L$ boards, while if Spoiler plays his 3rd round from $L$, Duplicator can maintain an isomorphism with any move played on either the 1st or 3rd board. We are left to consider just the possibilities that Spoiler plays $L(3)$ or $L(4)$ in board 1. Suppose he plays $L(3)$. Duplicator can then win by playing $B(5)$: if Spoiler plays his 3rd move from $B$ then $B(4)$ is met with $L(2)$ on the 1st board and any other $B$ move is easily parried on the 3rd $L$ board. On the other hand, if Spoiler plays his 3rd move from $L$ then whatever he does on the 3rd $L$ board can be matched with an isomorphism-preserving move on $B$. In the final case where Spoiler plays $L(4)$ for his 2nd round move on the 1st $L$ board, Duplicator responds with $B(5)$, guaranteeing an isomorphism. It follows that Duplicator can always win the $|B| = 5, |L| = 4$, 3-round game.

To complete the argument we must show that Duplicator can also win when one or both of $|B| > 5, |L| > 4$. These cases are easier and also covered in [8, Appendix B]. □

At this point it is natural to suspect that one can build up upper bounds recursively in a relatively simple manner. However, such an approach runs into unexpected difficulties, as we describe in the next section.

## 5. Interlude: Why Naive Recursion cannot be used to Build up Duplicator Winning Strategies in MS Games

It is worth pausing to understand why a simple idea to use recursion to build up Duplicator-winning strategies, and hence upper bounds on $g(r)$, fails. Via Lemma 4.2, we have established the fact that $g(3) \leq 4$ by showing that Duplicator wins 3-round MS games if the sizes of both linear orders are 4 or larger. It is tempting to try to use this fact to produce a Duplicator strategy for winning 4-round games using recursion. To understand the problematic logic, it suffices to consider a 4-round game on boards of sizes 9 and 10. The erroneous argument runs as follows: Suppose Spoiler plays $L(5)$ on his 1st move. Duplicator can then simply reply with the single move $B(5)$ (so the erroneous reasoning goes), as in Figure 8. Since there are five unplayed elements to the right of $B(5)$ and four unplayed elements to the right of $L(5)$, Duplicator should now just be able to mimic Spoiler's moves at, or to the left of, $B(5)/L(5)$ and otherwise play moves to the right of $B(5)/L(5)$ as if it were a 3-round, 5 versus 4 game, which we know is winnable by Duplicator. In fact, as we learned in Lemma 3.4, the 10 vs. 9 game is winnable by *Spoiler*. Hence this strategy does not work.


**Fig. 8.** A simple attempt to arrive at a Duplicator-winning strategy for the 10 versus 9 game. The first round moves are given in red.

The reason the strategy doesn't work is that there is interaction between play on the two sides and there are moves to the left of the 5 vs. 4 sub-game that are more powerful for Spoiler (in the sense of breaking more to-that-point maintained partial isomorphisms) than any moves available in the 5 vs. 4 game. This additional power is achieved by Spoiler playing atop an already played move that is not part of the 5 vs. 4 sub-game at a critical juncture. [8, Appendix C] provides additional details.

## 6. Games with Atoms: A different Approach to Obtaining Upper Bounds on $g(r)$

Let us define a new type of game. These games are similar to the MS games but with a twist. They will make it harder for Duplicator to win any particular game of $r$ rounds. Since Duplicator-winning strategies for a game of $r$ rounds provide upper bounds on $g(r)$, we will obtain upper bounds that at first look weaker, but will later match the lower bounds for $r \geq 4$, proving the upper bounds to be tight. The reason for considering these games is that they will allow us to recurse, getting around the issue we got stuck on in Section 5.

In our new game of $r$ rounds, we again have sets $\mathcal{A}$ and $\mathcal{B}$ of structures. However, each board on each of the sides, in addition to containing a structure $S$, contains a collection of unrelated, but labeled elements, $\{a_1, ..., a_s\}$, where $s$ can be as large as Spoiler wants. We shall refer to the collection of unrelated labeled elements as *atoms*. By a slight abuse of notation, we will treat atoms as if they are constant symbols, so that they can appear in sentences. However, these symbols will not appear in the sentences guaranteed by Theorems 1.2 and Theorem 1.3. These atoms are both unrelated to elements of the structure and unrelated to each other. On their respective turns, Spoiler and Duplicator play, as in the MS games, on all boards on their chosen side for that turn, picking an element from that board, choosing either an element from the structure, or from the set of atoms. If atom $a_k$ is selected by Spoiler on a given board, in a given round $j$, the only way Duplicator can maintain a partial isomorphism between that board and an opposite board is to select $a_k$ in round $j$ on the opposite board. When Duplicator makes copies of boards, the atoms are copied as well as the structures labeled with the moves made thus far.

In our figures, rather than showing all the atoms, and distinguishing those that are selected, we show only the atoms that have thus far been selected for a given board adjacent to the structures, with appropriate labeling. Let us refer to these new games as *MS games with atoms.*

**Observation 6.1.** *MS games with atoms are equivalent to MS games with structures that are a union of the prior structures and the set of atoms. Hence, by Theorem 1.2, we have that Spoiler wins $r$-round MS games with atoms on two sets of structures iff there exists a sentence $\Phi$ with up to $r$ quantifiers that distinguishes the two sets of structures. The sentence can contain constants and utilize an additional unary relation $A(x)$, which is true iff $x$ is an atom. Since the constants are constrained to only designate atoms, $A(c)$ holds for every constant $c$.*

Since a Spoiler winning strategy for a MS game is still a winning strategy for the analogous game with atoms (where atoms are never selected by Spoiler) we have:

**Lemma 6.2.** *If Spoiler can win an $r$-round MS game on a given pair of sets of structures then he can also win an $r$-round MS game with atoms on the same pair of sets of structures.*

**Corollary 6.3.** *If Duplicator can win an $r$-round MS game with atoms on a given pair of sets of structures, then she can also win an $r$-round MS game on the same pair of sets of structures.*

To understand the additional power provided to Spoiler by the atoms, let us revisit how Duplicator survives in one critical juncture of the 3-round, 5 versus 4 MS game from Lemma 4.2. Figure 9 shows the boards after two rounds in the critical sequence. In the 1st round, Spoiler played $B(3)$ and Duplicator played all possible moves on $L$. We consider the case where Spoiler then replies with 2nd round moves on $L$ as indicated in blue in the figure, and in particular playing on top of $L(1)$ on the board labeled $L_1$ and the board labeled $L_2$. Duplicator then replies with all possible responses on $B$ (again in blue). To the left of each $B$ board we indicate which $L$ boards the board has managed to keep an isomorphism with.

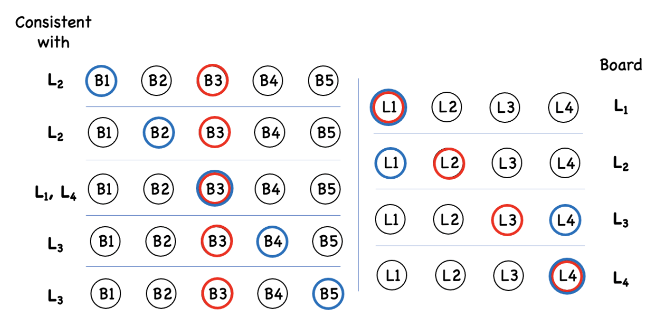


**Fig. 9.** Multi-structural game play on B versus L. 1st round moves are in Red, 2nd round moves in blue.

The crux of the matter is that because the third $B$ board is consistent with both $L_1$ and $L_4$, Spoiler cannot break both isomorphisms in the one remaining move.

With atoms, however, investing a turn to play an atom can usefully separate these two $L$ boards. Figure 10 illustrates

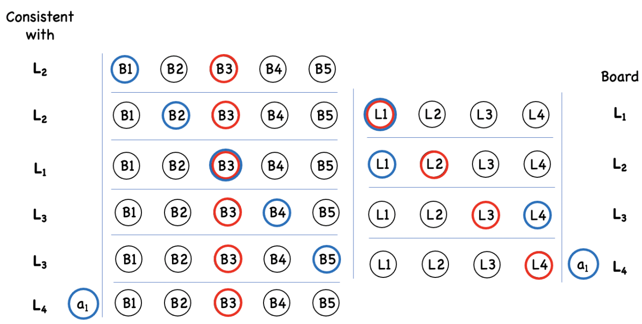


**Fig. 10.** Multi-structural game play with atoms.

Spoiler changing his second move on board $L_4$ from $L(4)$ to $a_1$. Duplicator now has an additional potentially viable 2nd round move, playing the newly played atom, $a_1$, as indicated in the additional board added on the left. However, note in the left hand column of the figure how no board is any longer simultaneously partially isomorphic with boards $L_1$ and $L_4$. Spoiler can now break each of the isolated isomorphisms by playing on $B$, in order, from top to bottom, as follows: $B(2), B(1), B(2), B(5), B(4), B(4)$.

The following follows from the discussion above and Lemma 4.2.

**Proposition 6.4.** *While Duplicator can always win* 3*-round MS games on linear orders of sizes* 5 *vs.* 4*, Spoiler can always win* 3*-round MS games with atoms on linear orders of these sizes.*

We are now going to establish Duplicator-winning strategies for MS games with atoms, for various numbers of rounds. By Lemma 6.2, Duplicator has a harder time winning these games than standard MS games and so these strategies will provide weaker upper bounds than the upper bounds obtained by providing Duplicator-winning strategies in MS games.

**Definition 6.5.** *Let $g'(r)$ denote the largest $k$ such that Spoiler wins every $r$-round MS game with atoms on two sets $\mathcal{B}$ and $\mathcal{L}$ of linear orders where each $B \in \mathcal{B}$ is of size at least $k$ and each $L \in \mathcal{L}$ is of size less than $k$.*

To see that $g'(r)$ is well-defined, note that if Duplicator can win an $r$-round E-F game on two linear orders $B, L$ then she can win an $r$-round MS game, as well as an $r$-round MS game with atoms, on sets $\mathcal{B}, \mathcal{L}$ of structures with $B \in \mathcal{B}$ and $L \in \mathcal{L}$ – this is the case because she can focus on the one pair of structures, $L$ and $B$, when playing the MS game, and further, because Spoiler gains no advantage from playing atoms when play is constrained to a single set of structures. Hence $g'(r) \leq f(r)$. In our considerations of $g'$ we will exclusively be focusing on game-based approaches. We will, further, just be concerned with establishing upper bounds on $g'(r)$, and in so doing will always be taking $\mathcal{B}$ and $\mathcal{L}$ to be singletons, $\mathcal{B} = \{B\}, \mathcal{L} = \{L\}$.

The following is an immediate consequence of Lemma 6.2.

**Lemma 6.6.** *For all values of $r$, we have $g(r) \leq g'(r)$.*

The following lemma describes why it is possible to recursively prove upper bounds on $g'$ (and hence $g$) in MS games with atoms and get around the issue described in Section 5.

**Lemma 6.7.** ***Reduction Lemma:*** *Suppose we have an $r$-round MS game with atoms on linear orders of sizes $K$ and $K'$. For some integer $h$, with $1 \leq h \leq \min(K, K')$, suppose there are boards on the $L$ and $B$ sides in which first moves of $L(h)$ and $B(h)$ have been played, or in which moves of $L(K-h+1)$ and $B(K'-h+1)$ have been played. Then Duplicator wins the $K$ vs. $K'$ game on those boards iff she wins the $(r-1)$-round MS game with atoms on linear orders of sizes $K-h$ and $K'-h$.*

Figure 11 illustrates the reduction of a 3-round, 6 versus 5 MS game with atoms, to a 2-round, 3 versus 2 MS game with atoms, per the conclusion.


**Fig. 11.** An illustration of the reduction in Lemma 6.7 where a 3-round, 6 versus 5 MS game with atoms gets reduced to a 2-round, 3 versus 2 MS game with atoms.

*Proof.* Without loss of generality assume the 1st round moves are $L(h)$ and $B(h)$. Duplicator now makes a pact with Spoiler, saying that the only way she will maintain an isomorphism with moves on the left hand side of $L(h)/B(h)$ is by mirroring (in other words by responding to $L(i)$ with $B(i)$ when $1 \leq i \leq h$, and vice versa). She tells Spoiler that if ever such a move is *not* mirrored, Spoiler can count it as a break in the partial isomorphism between the two boards. By agreeing to these stricter isomorphism rules, Duplicator makes it harder for herself to maintain partial isomorphisms. However, the effect is that we can remove the elements $B(1), ..., B(h)$ and $L(1), ..., L(h)$ and set aside the first $h$ atoms on each side, $a_1, ..., a_h$, only to be played as follows: if Spoiler ever would have wanted to play $B(i)$ or $L(i)$ on a given board, he can instead play $a_i$ with the same effect. This reduces the game to an $(r-1)$-round MS game with atoms on linear orders of size $K-h$ and $K'-h$ with $h$ additional atoms. Since in the definition of MS games with atoms, Spoiler already has as many atoms as he wants, the additional $h$ atoms have no effect on the game. We are left with just playing an $(r-1)$-round MS game with atoms on linear orders of size $K-h$ and $K'-h$ and if Duplicator can win such a game, she can win the original game.

On the other hand, if Duplicator does *not* have a winning strategy in the $K-h$ vs. $K'-h$ MS game with atoms, then

Spoiler has a winning strategy and can force play to be entirely on the side of the initial move with these many unplayed elements, and thus win. The lemma follows. □

**Lemma 6.8.** ***Laddering Up Lemma for MS Games and MS Games with Atoms:*** *Suppose Duplicator can win MS games (respectively MS games with atoms) on boards of sizes $K, K+1$ for all $K \geq N$. Then Duplicator can also win MS games (respectively MS games with atoms) on boards of sizes $K, K'$ for arbitrary $K, K' \geq N$.*

*Proof.* Suppose $K, K' \geq N$. Let l.o.($K$) denote the linear order of size $K$. Then we have that l.o.($K$) $\equiv_r$ l.o.($K+1$) $\equiv_r \cdots \equiv_r$ l.o.($K'$). By repeated application of Lemma 2.2 the lemma follows for MS games.

The same argument works for MS games with atoms by replacing each linear order with the union of the linear order and the corresponding atoms, whereby MS games with atoms reduce to MS games, per Observation 6.1. □

**Lemma 6.9.** *Duplicator can win 2-round multi-structural games with atoms on linear orders of sizes 2 or greater and hence $g'(2) \leq 2$.*

*Proof.* Immediate from Lemma 4.1 coupled with the observation that it never helps Spoiler to play an atom in the first or last round. □

**Lemma 6.10.** *Duplicator can win 3-round MS games with atoms on linear orders of sizes 5 or greater and hence $g'(3) \leq 5$.*

*Proof.* Suppose we have linear orders of sizes $K, K+1$ with $K \geq 5$. Any Spoiler 1st round move leaves a short side of no more than $\lfloor \frac{K}{2} \rfloor$ unplayed elements on that side. Duplicator can then reply with a move leaving an identical short side. Without loss of generality assume this common short side is on the left. Then to the right of the played moves, each board has at least $K - \lfloor \frac{K}{2} \rfloor - 1 \geq 2$ unplayed elements, and by virtue of the fact that $g'(2) \leq 2$ (Lemma 6.9), Lemma 6.7 guarantees that Duplicator has a winning strategy. The Laddering Up Lemma 6.8 then guarantees that Duplicator has a winning strategy for any boards of sizes $K, K' \geq 5$. The lemma follows. □

Although this section is concerned with establishing upper bounds on $g$, we shall actually need to establish precise values for $g'$ in order to get the upper bounds on $g$ to go through. The discussion we gave to show that Spoiler can win a 5 vs. 4 3-round MS game with atoms (Proposition 6.4) can easily be extended to show that Spoiler can win such a 3-round game on linear orders of sizes 5 or greater versus 4 or smaller. Hence we have that $g'(3) \geq 5$, so that together with the prior lemma we have:

**Lemma 6.11.** $g'(3) = 5$.

Any Spoiler-winning strategy in an ordinary MS game of $r$ rounds corresponds to a formula that is valid for $B$ but not for $L$ with $r$ quantifiers. If Spoiler's strategy starts on $B$ then the corresponding formula starts with $\exists$, while if Spoiler's strategy starts on $L$, the formula starts with $\forall$. If we force Spoiler to play first on $L$, then we are giving Duplicator an advantage so that she may be able to win games that she would not be able to win otherwise.

**Definition 6.12.** *Let $g'_\forall(r)$ denote the smallest $k$ such that Duplicator can win MS games with atoms on a pair of linear orders, each of size $k$ or greater, if Spoiler is constrained to play his first move on $L$.*

**Lemma 6.13.** *If Spoiler is constrained to play his first move from L, Duplicator can win $r$-round MS games with atoms on linear orders of sizes $2g'(r-1)$ or greater. Hence, $g'_\forall(r) \leq 2g'(r-1)$.*

*Proof.* Consider an $r$-round, $2g'(r-1)+1$ vs. $2g'(r-1)$ MS game with atoms. Refer to Figure 12. If Spoiler is constrained

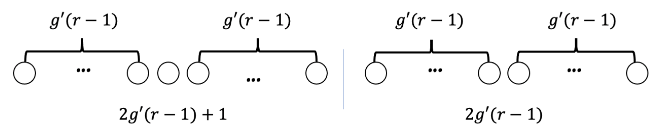


**Fig. 12.** The $r$-round, $2g'(r-1)+1$ vs. $2g'(r-1)$ MS game with atoms. Only one board on each side are shown.

to play on the $L$ side, his play will necessarily leave a short side of size at most $g'(r-1)-1$, which can be matched by a move that leaves the same short side on $B$, and with long sides that are each of size at least $g'(r-1)$. Hence the game is winnable by Duplicator via the Reduction Lemma (Lemma 6.7). For $r$ round games on boards of sizes $K+1$ vs. $K$ with $K > 2g'(r-1)$ again the short sides can be matched up, leaving long sides of sizes still at least $g'(r-1)$. The lemma follows by the Laddering Up Lemma (Lemma 6.8). □

**Theorem 6.14.** *For $r \geq 2$,*

$$g'(r) = \begin{cases} 2g'(r-1) & \text{if } r \text{ is even,} \\ 2g'(r-1)+1 & \text{if } r \text{ is odd.} \end{cases} \quad (10)$$

*Moreover, Duplicator can win $r$-round MS games with atoms on linear orders of sizes $2g'(r-1)$ or greater if $r$ is even, and on linear orders of sizes $2g'(r-1)+1$ or greater if $r$ is odd.*

*Proof.* We establish the equality asserted in the theorem via first establishing that the $\geq$ inequality holds, and then establishing that the $\leq$ inequality holds. Since for all $r \geq 1$ we have $g'(r) \geq g(r)$ (lemma 6.6), the $\geq$ portion of the theorem follows from the lemmas establishing that $g(1) = 1, g(2) \geq 2, g'(3) = 5$ and $g(4) \geq 10$, together with Theorem 3.5.

Now let us establish that the $\leq$ inequality holds. It is trivial to verify that $g'(1) = 1$, while $g'(2) \leq 2$ is Lemma 6.9 and $g'(3) = 5$ is Lemma 6.11. Hence, we have already established the $\leq$ part of the theorem for $r = 2$ and $r = 3$. For larger values of $r$, the case of odd $r$ is easy, so let's dispose of that case first – it is essentially the same argument we gave to establish $g'(3) \leq 5$ in Lemma 6.10. Suppose we have linear orders of sizes $K, K+1$ with $K \geq 2g'(r-1)+1$. Any

Spoiler 1st round move leaves a short side of no more than $\lfloor \frac{K}{2} \rfloor$ unplayed elements on that side. Duplicator can then reply with a move leaving an identical short side. Without loss of generality assume this common short side is on the left. Then, to the right of the played moves, each board has at least $K - \lfloor \frac{K}{2} \rfloor - 1$ unplayed elements. But $K - \lfloor \frac{K}{2} \rfloor - 1 \geq K - \frac{K}{2} - 1 \geq (g'(r-1) + \frac{1}{2}) - 1$, whence $K - \lfloor \frac{K}{2} \rfloor - 1 \geq g'(r-1) - \frac{1}{2}$. Since both $K - \lfloor \frac{K}{2} \rfloor - 1$ and $g'(r-1)$ are integers, it follows that $K - \lfloor \frac{K}{2} \rfloor - 1 \geq g'(r-1)$. Thus, each board has at least $g'(r-1)$ unplayed elements on their long sides. The Reduction Lemma (Lemma 6.7) in conjunction with the induction hypothesis therefore guarantees that Duplicator has a winning strategy. The Laddering Up Lemma (Lemma 6.8) then guarantees that Duplicator has a winning strategy for any boards of sizes $K, K' \geq 2g'(r-1) + 1$.

Next consider the case of even $r$. Suppose first that we have linear orders of sizes $2g'(r-1)$ and $2g'(r-1)+1$. Let us first dispose of any first move by Spoiler other than $B(g'(r-1)+1)$, the middle element on the $B$ side. Any move other than this one on the $B$ side can be responded to with a move on $L$ that matches the short side while leaving at least $g'(r-1)$ unplayed elements on both long sides, and therefore, again by the Reduction Lemma and induction, yielding a winning strategy for Duplicator. On the other hand, any Spoiler 1st round move on $L$ is analogously met by matching the short side with a move on $B$, transposing to the just-prior analysis. Thus we may assume that Spoiler plays the element $B(g'(r-1) + 1)$. In response, Duplicator uses two boards and plays $L(g'(r-1))$ on one of the boards and $L(g'(r-1) + 1)$ on the other one. See Figure 13.

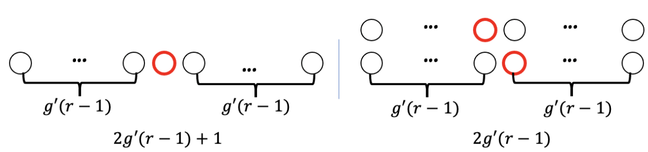


**Fig. 13.** A $2g'(r-1)+1$ vs. $2g'(r-1)$ game where Spoiler plays first on $B(g'(r-1)+1)$ and Duplicator responds by playing $L(g'(r-1))$ on one board and $L(g'(r-1)+1)$ on another.

Consider the possible Spoiler 2nd round responses. Suppose Spoiler plays on $B$. A play on one of the left-hand unplayed $g'(r-1)$ elements, i.e., on some $B(i)$ such that $1 \leq i \leq g'(r-1)$ will be met with a play of $L(i)$ on the 2nd $L$ board. As we argued in the proof of the Reduction Lemma, we can now regard the remaining $g'(r-1) - 1$ unplayed elements to the left of the 1st round selections on the bottom $L$ board and the $B$ boards as additional atoms and just consider the game on the right side of these boards, which is now an $r-2$ round game on boards of sizes $g'(r-1)$ and $g'(r-1) - 1$. Inductively it is easy to see that $g'(r-1) - 1 > g'(r-2)$ and so such a sequence leads inductively to a Duplicator win. A 2nd round Spoiler play on $B$, on one of the right hand set of unplayed $g'(r-1)$ elements, is handled with a symmetrical argument. Further, playing an atom on $B$ is met by playing the same atom on both $L$ boards (in fact playing on just one board and ignoring the other is sufficient), and again leads to an inductive win for Duplicator by virtue of the fact that $g'(r-1) - 1 > g'(r-2)$.

Thus we may assume that Spoiler makes his 2nd round moves on $L$. A play on the long side of either board is met with a symmetrical play by Duplicator on $B$, transposing to our prior analysis for when Spoiler played his 2nd move on $B$. Playing an atom on $L$ is met with the same atom being played on $B$, again with a transposition. Thus we may suppose that Spoiler plays on the short side of both $L$ boards, and, in particular, plays on the left on the top $L$ board. Now the left hand (short side) of the top board is of size $g'(r-1) - 1$ and the left hand side of $B$ is of size $g'(r-1)$. Further, we are assuming that $r$ is even, and so $r-1$ is odd and hence by (10) we have

$$g'(r-1) = 2g'(r-2) + 1. \tag{11}$$

Since we've reduced the analysis to Spoiler next playing on the $L$ side of this $(r-1)$-round, $g'(r-1)$ vs. $g'(r-1)-1$ game, Lemma 6.13 applies and says that $g'_{\forall}(r-1) \leq 2g'(r-2)$. Taken together with (11), we have that $g'_{\forall}(r-1) \leq g'(r-1) - 1$.

Thus, we are left with boards of sizes $g'(r-1)$ and $g'(r-1) - 1$, both of which are at least of the size of $g'_{\forall}(r-1)$. Hence, by the definition of $g'_{\forall}$ (Definition 6.12), Duplicator has a winning strategy, over the remaining $r-1$ rounds, playing just on the left hand sides of these two boards and hence, by the Reduction Lemma, has a winning strategy playing on the entire board. Thus Duplicator has a winning strategy in the original $r$-round game for boards of sizes $2g'(r-1)$ and $2g'(r-1) + 1$.

For $K, K+1$ with $K > 2g'(r-1)$ the argument is easier since Duplicator can just mimic the short side play of any 1st round Spoiler play and immediately apply the Reduction Lemma. As usual, the argument is completed by applying the Laddering Up Lemma. □

We have therefore established the following upper bounds:

**Corollary 6.15.** *We have $g(2) \leq 2, g(3) \leq 4, g(4) \leq 10$, and for $r > 4$,*

$$g(r) \leq \begin{cases} 2g(r-1) & \text{if } r \text{ is even,} \\ 2g(r-1)+1 & \text{if } r \text{ is odd.} \end{cases}$$

*Moreover, Duplicator can win $r$-round MS games on linear orders of sizes that are at least as large as these upper bound (right hand side) values in all the inequalities.*

*Proof.* The inequalities $g(2) \leq 2, g(3) \leq 4$ are Lemmas 4.1 and 4.2. The chain of inequalities $g(4) \leq g'(4) \leq 2g'(3) \leq 10$, follows from Lemma 6.6, Lemma 6.10 and Theorem 6.14. The inductively defined inequality for $r > 4$ follows by Lemma 6.6 and Theorem 6.14. Finally, the upper bounds associated with all these lemmas and one theorem were established and stated by observing that Duplicator could win $r$ round games when both linear orders were at least as large as the upper bounds. The same therefore follows for this corollary. □

## 7. Tight Bounds on $g(r)$ and Final Theorem

At last, we pull together the identical upper and lower bounds we have obtained for $g(r)$, to yield Theorem 1.6.

Although we have a completely specified $g(r)$, we have not completely answered the question of the minimum number of quantifiers needed to distinguish one linear order, $B$, from another, $L$. Specifically, if $g(r-1) \leq |L| < |B| < g(r)$ for some value of $r$, we know there is no formula with $r-1$ quantifiers that distinguishes $B$ from $L$, but nothing more. The following lemma closes this gap.

**Lemma 7.1.** *Given two linear orders $B, L$, with $|L| < |B| < g(r)$ for some $r > 0$ then there is a formula with $r$ quantifiers that distinguishes $B$ from $L$.*

For a proof of this lemma, which mostly entails taking a closer look at Theorem 3.5, see [8, Appendix E].

Finally, we are able to prove the precise game theoretic analog of Theorem 1.7.

**Theorem 7.2.** *Two linear orders, $B$ and $L$, with $|L| < |B|$, can be distinguished by a formula with $r$ quantifiers iff $|L| < g(r)$.*

*Proof.* If direction: If $|L| < |B| < g(r)$ then $B$ and $L$ can be distinguished by Lemma 7.1. If $|L| < g(r) \leq |B|$ then $B$ and $L$ can can be distinguished by the definition of $g$ (Definition 1.5).

Only if direction: If $g(r) \leq |L| < |B|$ then $B$ and $L$ *cannot* be distinguished by Corollary 6.15. □

## 8. Conclusions

We have studied multi-structural games, which generalize E-F games by being played over sets $\mathcal{A}, \mathcal{B}$ of structures rather than over a pair $A$, $B$ of individual structures. Whereas E-F games can capture exactly the quantifier rank needed to describe a property, we showed that multi-structural games can capture exactly the number of quantifiers needed to describe a property. As a first application, we used them to determine the number of quantifiers needed to distinguish between linear orders of different sizes.

The quantifier count is a natural complexity measure but has received scant attention compared to the quantifier rank, number of distinct variable names, and measures of size and depth for the formula body. We expect complexity differences to be magnified in studying other structures beyond linear orders, such as higher-dimensional lattices, rooted trees, and other classes of graphs. As with the related ideas of Lotfallah [19], MS games extend readily to second-order logic where they may bear on higher-order problems in descriptive complexity. For example, MS games when adapted to second order logic can easily simulate Ajtai-Fagin games.


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