COMPUTATIONAL AND COMBINATORIAL ASPECTS OF FINITE SIMPLE GROUPS

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SFB 878 – Groups, Geometry & Actions

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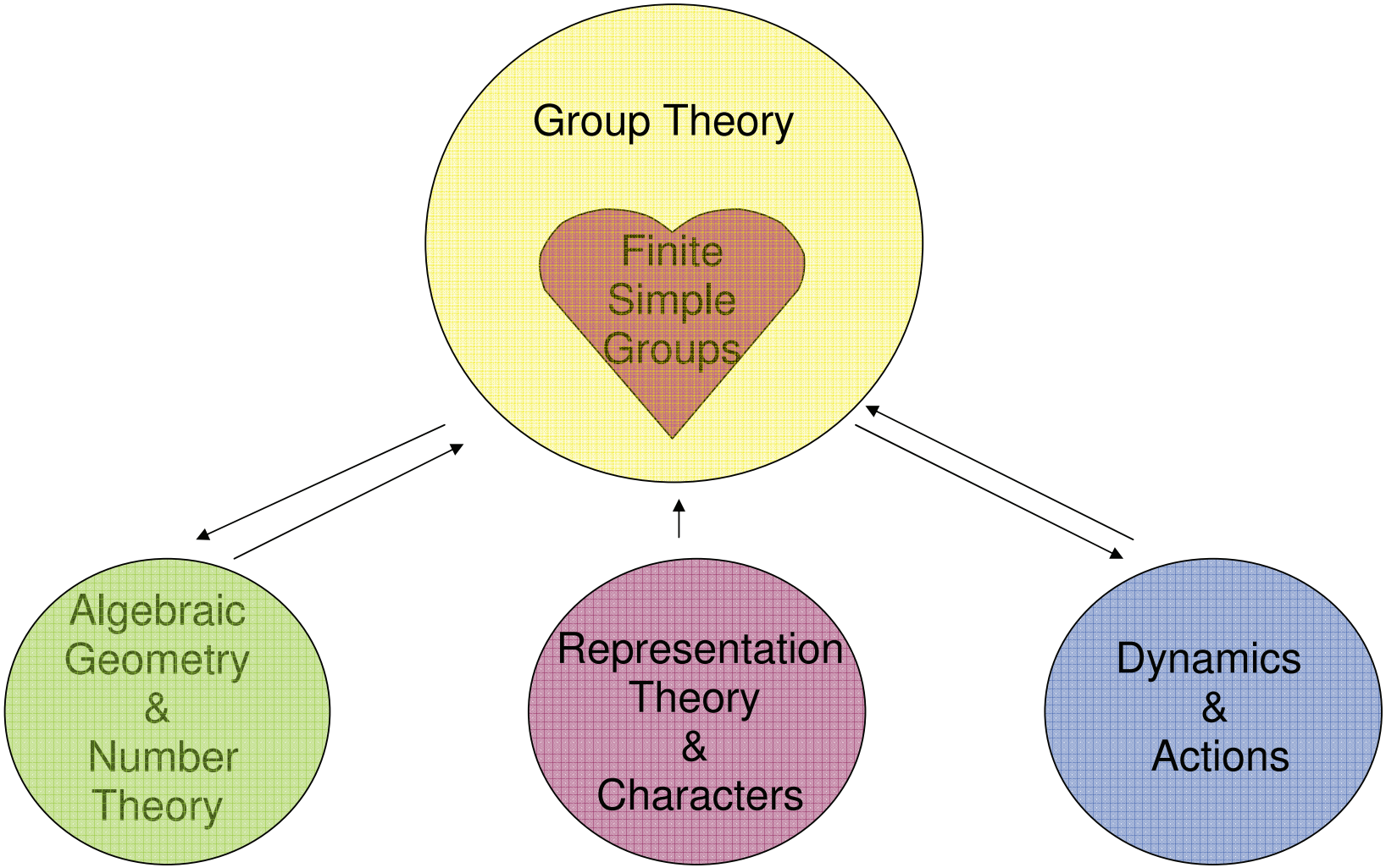
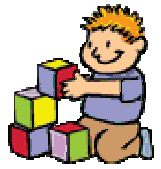

Group Theory
Finite
Simple
Groups
Algebraic
Geometry
&
Number
Theory
Representation
Theory
&
Characters
Dynamics
&
Actions

Finite simple groups



The finite simple groups are the building blocks of all finite groups.

Definition. G is simple if it has no non-trivial normal subgroups.

Theorem. Classification of the (non-abelian) Finite Simple Groups.

- Alternating groups A_n ($n \geq 5$).
- Finite simple groups of Lie type $G_r(q)$
  where r is the Lie rank and $q=p^e$ is the size of finite field; e.g. $PSL_{r+1}(q)$.
- 26 sporadic groups.

On the proof… thousands of pages, hundreds of articles, ~100 authors:
From Galois (1832) to Gorenstein-Lyons-Solomon (90's)…

The Product Replacement Algorithm

The problem

Basic problem in computational group theory:

How to generate a random element in a finite group G?

The Product Replacement Algorithm (PRA) was suggested in 1995 by Celler, Leedham-Green, Murray, Niemeyer & O'Brien.

The PRA showed very good performance in practical experiments, but there is no rigorous justification. It was included in GAP and MAGMA.

The PRA performs a random walk on the product replacement graph $\Gamma_n(G)$ whose vertices are the generating n-tuples of G (for a fixed n):

$$\{ (g_1,\ldots,g_n) : \langle g_1,\ldots,g_n \rangle = G \}$$

Question: Let G be a finite simple group. Is $\Gamma_n(G)$ connected?

The Product Replacement Algorithm

The results

n=2	n≥3
[Neumann, 1951]. $\Gamma_2(A_5)$ is disconnected. [Garion-Shalev, 2009]. (Conjectured in 2002 by [Guralnick-Pak]). If G is a finite simple group, then the number of connected components of $\Gamma_2(G)$ grows to infinity as $\lvert G\rvert\to\infty$.	Wiegold's conjecture (1980's). If G is a finite simple group and n≥3 then $\Gamma_n(G)$ is connected. $\Gamma_n(G)$ is connected: • [Gilman, 1977]. $G=PSL_2(p)$, n≥3. • [Evans, 1993]. $G=PSL_2(2^e)$, $Sz(2^{2e+1})$, n≥3. • [Garion, 2008]. $G=PSL_2(p^e)$, n≥4. • [Avni-Garion, 2008]. $G=G_r(p^e)$, n≥c(r), finite simple group of Lie type of Lie rank r.

The Product Replacement Algorithm

The methods

Is $\Gamma_n(G)$ connected?

$\Downarrow$

Can we connect any generating n-tuple of G to a redundant one?

$\Downarrow$

What are the subgroups of G?

[Garion, 2008]. For $G=PSL_2(q)$: the subgroups are well-known [Dickson, 1901].

[Avni-Garion, 2008]. For $G=G_r(q)$ – a finite simple group of Lie type:

- Aschbacher's classification of maximal subgroups (1984) – uses CFSG.
- [Larsen-Pink, 1998] – uses algebraic geometry (not CFSG!).

[Garion-Glasner]. For an infinite simple Tarski monster group G:

- Any subgroup is a cyclic group of order p (p is either a fixed prime or ∞).
- Theorem. A faithful highly transitive action of $Out(F_n)$ on a countable set.

Word maps in groups

The problem

A word $w=w(x_1,\ldots,x_n)$ is an element in $F_n=\langle x_1,\ldots,x_n\rangle$.

For a group G, a word map is given by:

$$w: G^n \to G$$

$$(g_1,\ldots,g_n) \to w(g_1,\ldots,g_n)$$

Questions: Let G be a finite simple group and let $w\neq 1$.

- What is the image $w(G^n)$? Is $w: G^n \to G$ surjective?
- What are the sizes of the fibers of a word map?
  Is $w: G^n \to G$ almost equidistributed? e.g. $|w^{-1}(g)| \approx |G|^{n-1}$ for almost all $g\in G$.

Investigated by Guralnick, Larsen, Liebeck, Segal, Shalev, O'Brien, Tiep, …

Word maps in groups

The results

Commutator word: $e_1 = [x,y] = xyx^{-1}y^{-1} \in F_2$

Theorem (Ore's Conjecture) [Ore, 1951; Thompson, 1960's; Ellers-Gordeev, 1998; Liebeck-O'Brien-Shalev-Tiep, 2008; …]

Any element in a finite simple group is a commutator.

[Garion-Shalev, 2009].

The commutator map on finite simple groups is almost equidistributed.

Engel words: $e_n = [e_{n-1},y] = [\ldots[[x,y],y],\ldots,y] \in F_2$

[Bandman-Garion-Grunewald]. Surjectivity and equidistribution of e_n on $PSL_2(q)$.

Two-power words: $w=x^a y^b \in F_2$ [Guralnick-Malle, 2012; LOST, 2012].

[Bandman-Garion, 2012]. Surjectivity and equidistribution of $x^a y^b$ on $PSL_2(q)$.

Word maps in groups

The methods

Commutator word:

[Frobenius, 1896]. $\#\{(x,y)\in G\times G: [x,y]=g\} = |G| \cdot \sum_{\chi\in Irr(G)} \chi(g)/\chi(1)$

Words in $SL_2(q)$:

Trace map Theorem [Fricke-Klein, 1897; Vogt, 1889].

word $w(x,y)$ in $SL_2(q)$ $\Rightarrow$ $tr(w)=P(s,t,u)$ is a polynomial in $s=tr(x)$, $t=tr(y)$, $u=tr(xy)$ over $\mathbf{F}_q$

Examples: $w=[x,y] \Rightarrow tr(w) = s^2 + t^2 + u^2 - stu - 2$

[Bandman-Garion-Grunewald]. $w=e_n(x,y) \Rightarrow s_n = tr(e_n) = s_{n-1}^2 + 2t^2 - s_{n-1}t^2 - 2$

[Bandman-Garion, 2012]. $w=x^a y^b \Rightarrow tr(w) = u\cdot f_{a,b}(s,t) + h_{a,b}(s,t)$

By induction, compute $tr(w)$ for any $w\in F_2$

Beauville surfaces

The problem

Beauville surface [Beauville, 1978; Catanese, 2000]. $S=(C_1\times C_2)/G$

S is an infinitesimally rigid complex surface, where C_1 and C_2 are curves of genus ≥ 2 and G is a finite group acting freely on their product.

Beauville structure [Bauer-Catanese-Grunewald, 2005]. $(x_1,y_1,z_1;x_2,y_2,z_2)$

- $x_1y_1z_1 = 1 = x_2y_2z_2$,
- $\langle x_1,y_1\rangle = G = \langle x_2,y_2\rangle$,
- no non-identity power of x_1,y_1,z_1 is conjugate in G to a power of x_2,y_2,z_2.

The type of $(x_1,y_1,z_1;x_2,y_2,z_2)$ is the 6 orders of $x_1,y_1,z_1;x_2,y_2,z_2$.

Questions [Bauer-Catanese-Grunewald, 2005].

1) Which finite simple groups admit a Beauville structure?

2) Which types can occur in a Beauville structure?

Beauville surfaces

The results

Conjecture 1 [BCG, 2005].
All finite simple groups (except A_5) admit a Beauville structure.

Proved for:

- A_n ($n \geq 6$) [BCG, 2005; Fuertes,Gonzáles-Diez, 2009].
- $PSL_2(q)$ ($q \geq 7$) [Fuertes-Jones, 2011; Garion-Penegini].
- **Almost all** finite simple groups [Garion-Larsen-Lubotzky, 2012].
- **All** finite simple groups ($\neq A_5$) [Fairbairn-Magaard-Parker; Guralnick-Malle].

Conjecture 2 [BCG, 2005] – proved by [Garion-Penegini].
For any two hyperbolic triples of integers $(k_1,l_1,m_1;k_2,l_2,m_2)$ almost all alternating groups A_n admit a Beauville structure of type $(k_1,l_1,m_1;k_2,l_2,m_2)$.

[Garion]. Characterization of the types of Beauville structures for $PSL_2(q)$.

Beauville surfaces

The methods

G admits a Beauville structure of type $(k_1,l_1,m_1;k_2,l_2,m_2)$.

$\Leftrightarrow$

G is a quotient of $\Delta(k_1,l_1,m_1)$ and $\Delta(k_2,l_2,m_2)$ + "disjoint" condition.

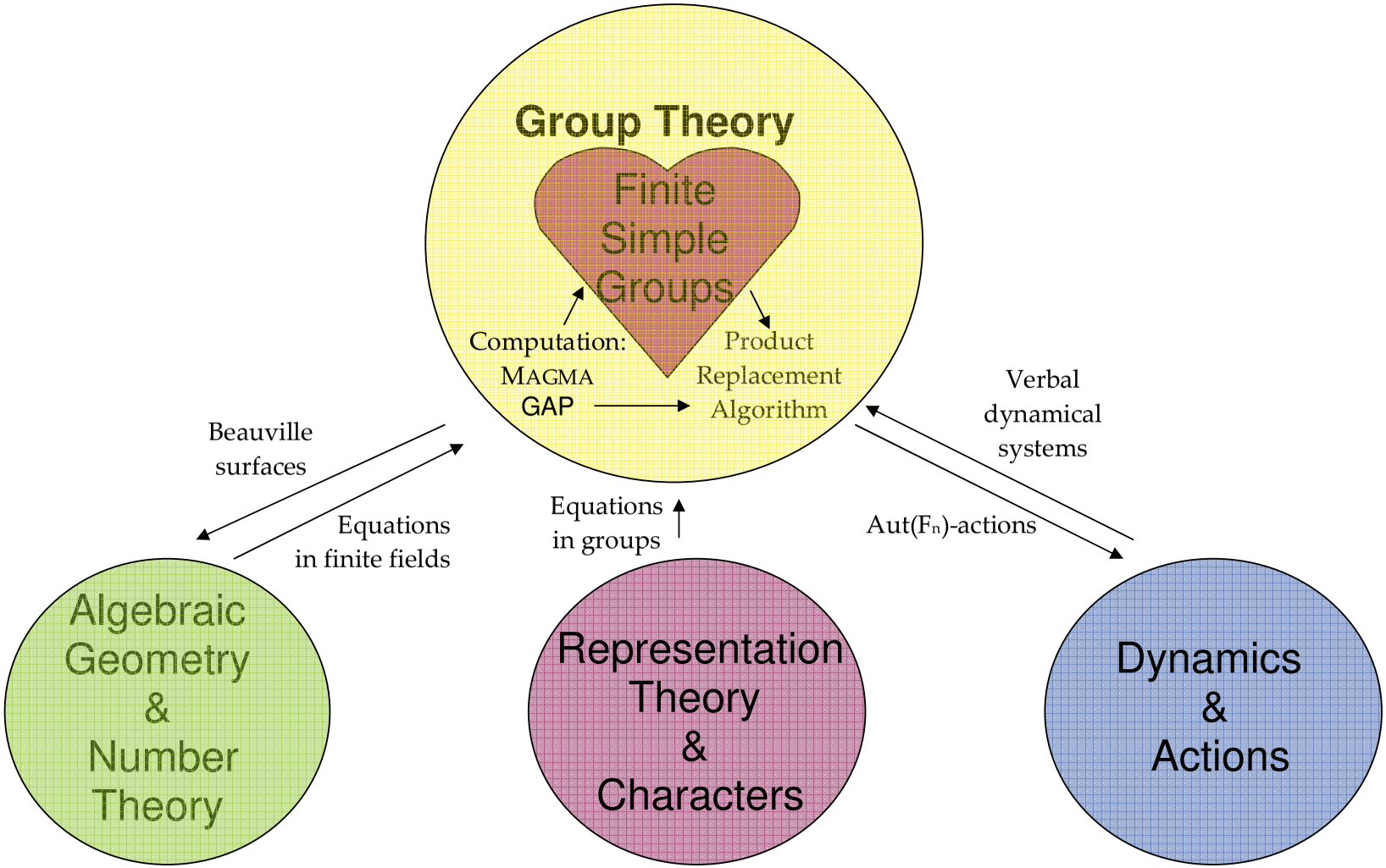
Triangle group: $\Delta(k,l,m) = \langle x,y: x^k = y^l = (xy)^m = 1 \rangle$

Question: Which finite simple groups are quotients of a given $\Delta(k,l,m)$?

- A_n – Higman 1960s; Conder 1980; Everitt 2000; Liebeck-Shalev 2004,…
- $PSL_2(q)$ – Macbeath 1968; Rosenberger et al. 1989; Marion 2009.
- $G_r(q)$ – open! Luccini-Tamburini-Wilson 2000; Liebeck-Shalev 2005,…

[Frobenius, 1890s]. X,Y,Z – conjugacy classes in G (of orders k,l,m).

$\#\{x,y,z: x\in X, y\in Y, z\in Z, xyz=1\} = |X|\cdot|Y|\cdot|Z|/|G| \cdot \sum_{\chi\in Irr(G)} \chi(x)\chi(y)\chi(z)/\chi(1)$


Group Theory
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Computation: MAGMA GAP
Product Replacement Algorithm
Beauville surfaces
Equations in finite fields
Equations in groups
Verbal dynamical systems
Aut(Fn)-actions
Algebraic Geometry & Number Theory
Representation Theory & Characters
Dynamics & Actions

Future plans

Continue my research in group theory, focusing on finite simple groups, while interacting with other fields of mathematics such as algebraic geometry, number theory, representation theory, dynamics…

Some specific research problems…

Word maps

- Analysis of general words in $PSL_2(q)$.
- Generalize the trace map method to $PSL_n(q)$ and $G_r(q)$.
- Generalizations to $SL_2(\mathbf{Z}_p)$ and $SL_2(\mathbf{Z})$.

Beauville surfaces

- What is the probability of admitting a Beauville structure?
- Constructing Beauville surfaces with specific properties (e.g. reality).
- Investigating the moduli space of Beauville surfaces.