BEAUVILLE SURFACES AND PROBABILISTIC GROUP THEORY

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Outline

- Triangle groups and Hurwitz groups:

  Which finite simple groups are quotients of given triangle groups?

- Beauville surfaces and Beauville structures:

  Which finite simple groups admit Beauville structures?

Triangle groups

$\Delta_{k,l,m} = \langle x,y : x^k = y^l = (xy)^m = 1 \rangle.$

Spherical

$1/k+1/l+1/m>1$

$\Delta_{k,l,m}$ is finite:

$\Delta_{2,2,m} = D_{2m}$

$\Delta_{2,3,3} = A_4$

$\Delta_{2,3,4} = S_4$

$\Delta_{2,3,5} = A_5$

Euclidian

$1/k+1/l+1/m=1$

$\Delta_{k,l,m}$ is infinite:

$\Delta_{3,3,3}$

$\Delta_{2,4,4}$

$\Delta_{2,3,6}$

"wall-paper" groups

Hyperbolic

$1/k+1/l+1/m<1$

$\Delta_{k,l,m}$ is infinite.

$\Delta_{2,3,7}$ is Hurwitz triangle group

$1-(1/k+1/l+1/m)$ is minimal for $(2,3,7)$:

$1-(1/2+1/3+1/7) = 1/42$

Finite quotients of triangle groups

Question: Which finite (simple) groups are quotients of $\Delta=\Delta_{k,l,m}$?

Triangle group Δ
$\downarrow$
Finite group
$\downarrow$
Finite simple group

Finite quotients of $\Delta_{2,3,7}$ are called Hurwitz groups.

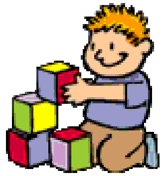
Example: $PSL_2(7)$ (order=168).

Motivation: Riemann surfaces [Hurwitz's automorphisms Theorem, 1893].

Question: Which finite simple groups are Hurwitz groups?

[Higman 60's, Macbeath '68, Conder, Cohen, Di-Martino, Luccini, Malle, Tamburini, Vincent, Vsemirnov, Wilson, Zalesski, …]

Finite simple groups



The finite simple groups are the building blocks of all finite groups.

Definition. G is simple if it has no non-trivial normal subgroups, except G.

Theorem. Classification of the (non-abelian) Finite Simple Groups.

- Alternating groups A_n $(n \geq 5)$.
- Finite simple groups of Lie type $G_r(q)$
  where r is the Lie rank and $q=p^e$ is the size of finite field; e.g. $PSL_{r+1}(q)$.
- 26 sporadic groups.

On the proof… thousands of pages, hundreds of articles, ~100 authors:
From Galois (1832) to Gorenstein-Lyons-Solomon (90's)…

Corollary. If G is a finite simple group then G is 2-generatred.
Hence, G is a quotient of some triangle group Δ.

Which finite simple groups are quotients of $\Delta_{k,l,m}$?

Higman's conjecture (60's):
A_n is a quotient of $\Delta_{k,l,m}$
if $n \geq n_0(k,l,m)$ is large enough.

- (2,3,7): [Higman; Conder '80]. $n_0=168$.
- (k,l,m): [Everitt '00]; [Liebeck-Shalev '04].

Conjecture [Liebeck-Shalev '05].
$G_r(q)$ is a quotient of $\Delta_{k,l,m}$
if $r \geq r_0(k,l,m)$ is large enough.

- (2,3,7): [Luccini-Tamburini '99; Luccini-Tamburini-Wilson '00]. For $PSL_r(q)$: $r_0=267$

For almost all primes p there is exactly one exponent e s.t. $PSL_2(p^e)$ (or: $PGL_2(p^e)$) is a quotient of $\Delta_{k,l,m}$.

- (2,3,7): [Macbeath '68].
- (k,l,m): [Langer-Rosenberger '89; Levin-Rosenberger '90; Marion '09].

[Larsen-Lubotzky-Marion].

If (k,l,m) is a *rigid* triple of primes for G_r then there are finitely many e for which $G_r(p^e)$ is a quotient of $\Delta_{k,l,m}$.

(via deformation theory).

Beauville surfaces and Beauville structures

Algebro-geometric definition:

[Catanese '00]

A Beauville surface S is a compact complex surface

$$S=(C_1\times C_2)/G,$$

where C_1 and C_2 are curves of genus at least 2, G is a finite group acting freely on their product, and S is rigid.

Example [Beauville '78].

$C_1=C_2=\{x^5+y^5+z^5=0\}$, $G=\mathbf{Z}_5^2$

$\Leftrightarrow$

Group theoretical property:

[Bauer-Catanese-Grunewald '05]

G admits a Beauville structure of type $((k_1,l_1,m_1),(k_2,l_2,m_2))$, if there exist $((x_1,y_1,z_1),(x_2,y_2,z_2))$ such that

i. $\langle x_1,y_1\rangle = G = \langle x_2,y_2\rangle$,

ii. $x_1y_1z_1 = 1 = x_2y_2z_2$,

iii. $\mathrm{ord}(x_i)=k_i$, $\mathrm{ord}(y_i)=l_i$, $\mathrm{ord}(z_i)=m_i$,

iv. $\Sigma_1\cap\Sigma_2 = \{1\}$, where:

Σ_i is the union of the conjugacy classes of all powers of x_i, all powers of y_i, and all powers of z_i.

Note: (i)+(ii)+(iii) $\Leftrightarrow$ Δ_{k_i,l_i,m_i} map onto G

Beauville structures for finite simple groups

Question 1. Which finite simple groups admit a Beauville structure?

Theorem (conjectured by [Bauer-Catanese-Grunewald '05]).
All finite simple groups (except A_5) admit a Beauville structure.

Proved for…

- A_n ($n \geq 6$) [BCG '05; Fuertes- Gonzáles-Diez '09].
- $PSL_2(q)$ ($q \geq 7$) [Fuertes-Jones '11; Garion-Penegini '13].
- Other groups of low Lie rank $r \leq 2$ [Fuertes-Jones '11; Garion-Penegini '13].
- **Almost all** finite simple groups [Garion-Larsen-Lubotzky '12].
- **All** finite simple groups [Guralnick-Malle'12; Fairbairn-Magaard-Parker'13].

Types of Beauville structures

Question 2. Which types can occur in a Beauville structure?

Theorem [Garion-Penegini '13] (conjectured by [BCG '05]).

For any hyperbolic triples $((k_1,l_1,m_1),(k_2,l_2,m_2))$, almost all A_n admit a Beauville structure of type $((k_1,l_1,m_1),(k_2,l_2,m_2))$.

Remark. The proof is based on [Liebeck-Shalev '04].

Theorem [Garion]. Characterization of the types $((k_1,l_1,m_1),(k_2,l_2,m_2))$ of Beauville structures for $PSL_2(q)$.

Conjecture [Garion-Penegini '13]. For any hyperbolic triples (k_1,l_1,m_1) and (k_2,l_2,m_2), $G_r(q)$ admits a Beauville structure of type $((k_1,l_1,m_1),(k_2,l_2,m_2))$ if the Lie rank $r \geq r_0(k_1,l_1,m_1,k_2,l_2,m_2)$ is large enough.

Sketch of the probabilistic proof

In order to show that a finite simple group G admits a Beauville structure one needs to find $x_1,y_1,z_1,x_2,y_2,z_2 \in G$ such that:

(1) No non-identity power of x_1,y_1,z_1 is conjugate in G to a power of x_2,y_2,z_2:

Choose "disjoint" conjugacy classes X_1,Y_1,Z_1,X_2,Y_2,Z_2 .

- $G=G_r(q)$: conjugacy classes of regular semisimple elements t_1,t_2 from "disjoint" maximal tori $T_1=C_G(t_1),T_2=C_G(t_2)$, i.e. $\forall g\in G$: $T_1 \cap g^{-1}T_2 g = \{1\}$.

- $G=A_n$: almost homogeneous conjugacy classes whose elements are of orders k_1,l_1,m_1,k_2,l_2,m_2, with different numbers of fixed points.

[Liebeck-Shalev '04]. An almost homogeneous class in S_n (of order m):

$$\underbrace{(*,*,\ldots,*)}_{m}\underbrace{(*,*,\ldots,*)}_{m}\ldots\underbrace{(*,*,\ldots,*)}_{m}\underbrace{**\ldots*}_{f}$$

(f is bounded as $n\to\infty$)

Sketch of the probabilistic proof - continued

(2) $x_1y_1z_1 = 1 = x_2y_2z_2$:

[Frobenius]. Let: $N_{X,Y,Z} = \{(x,y,z): x\in X,\ y\in Y,\ z\in Z,\ xyz=1\}$

Then $\#N_{X,Y,Z} = |X|\cdot|Y|\cdot|Z|\cdot|G|^{-1} \cdot \sum_{\chi\in \mathrm{Irr}(G)} \chi(x)\chi(y)\chi(z)\chi(1)^{-1}$.

Usually, the main contribution to this sum comes from $\chi=1$.

We should estimate $|\chi(g)|$ for an irreducible character χ and an element g.

Claim. There exist an absolute constant c such that for every sufficiently large group of Lie type $G=G_r(q)$, there exist maximal tori T_1,T_2, such that for every regular $t\in T_1\cup T_2$ and every irreducible character χ of G: $|\chi(t)|<cr^3$.

Witten's zeta function: $\zeta^G(s) = \sum_{\chi\in \mathrm{Irr}(G)} \chi(1)^{-s}$

[Liebek-Shalev '05]. If $G\neq PSL_2(q)$ and $s>2/3$ then

$$\zeta^G(s)\to 1 \text{ as } |G|\to\infty .$$

Sketch of the probabilistic proof - continued

(3) $<x_1,y_1> = G = <x_2,y_2>$:

Show that N_{X_i,Y_i,Z_i} contains a generating pair (x_i,y_i) for G.
Namely, one should avoid pairs which lie in the same maximal subgroup.

[Dixon '69; Kantor-Lubotzky '90; Liebeck-Shalev '95].
Almost every pair $(x,y)\in G\times G$ in a finite simple group G generates it.

Estimate the cardinality of N_{X_i,Y_i,Z_i} (using Frobenius), and compare it to:

$$\Sigma_{M\in \max G} \{(x,y,z): x\in X_i\cap M,\ y\in Y_i\cap M,\ z\in Z_i\cap M,\ xyz=1\} \leq$$
$$\Sigma_{M\in \max G} |M|^2 \leq |G|^2\, \Sigma_{M\in \max G}[G:M]^{-2} \leq |G|^2\, m(G)^{-1/2}\, \Sigma_{M\in \max G} [G:M]^{-3/2}$$

$\max G$ = set of proper maximal subgroups of G.
$m(G)$ = minimal index of a proper subgroup of G.
= minimal degree of a non-trivial permutation representation of G.

$m(G) \geq cq^r$ [Landazuri-Seitz '74].

[Liebeck-Martin-Shalev '05]. $\forall s>1,\ \Sigma_{M\in \max G} [G:M]^{-s} \to 0$ as $|G|\to\infty$.

The probability of admitting a Beauville structure

Question 3. What is the probability $P(G)$ that 4 random elements (x_1,y_1,x_2,y_2) in a finite simple group G constitute a Beauville structure?

Conjecture.

- $P(A_n) \to 1$ as $n \to \infty$.
- $P(G_r(q)) \to 1$ as $r \to \infty$.

Observation. $P(G_r(q))$ is bounded above by a function of the Lie rank r.

Theorem [Garion]. $1/32-\varepsilon < P(PSL_2(q)) < 35/36+\varepsilon$, where $\varepsilon=\varepsilon(q) \to 0$ as $q \to \infty$.