MODELLING AND ANALYZING BREAKDOWN PHENOMENA IN INSULATORS - A STOCHASTIC APPROACH

Research Thesis

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Science

EMMANUEL YASHCHIN

MODELLING AND ANALYZING BREAKDOWN PHENOMENA IN INSULATORS - A STOCHASTIC APPROACH

Research Thesis

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Science

EMMANUEL YASHCHIN

Submitted to the Senate of the Technion - Israel Institute of Technology

Iyar 5741 . H a i f a May 1981

This research was carried out under the supervision of Dr. Paul Feigin in the Faculty of Industrial Engineering and Management.

I would like to thank Dr. Paul Feigin for his ideas and good advice which made the present research possible.

Special thanks are due to Professor Nicolas Klein (Faculty of Electrical Engineering, Technion) for motivating this research and for his invaluable consultations.

My sincere thanks are also due to Professor J.L. Teugels (The Catholic University of Louvain) for his help and good advice and the referees, for their valuable comments.

Thanks are also due to Mrs. Hadassah Nesher for her excellent work in the preparation of the manuscript.

C O N T E N T S

	Page No.
Abstract	1
CHAPTER I	3
1. Introduction	3
1.1 The breakdown phenomenon	5
1.1.1 Description	5
1.1.2 Data available	9
1.2 Some physical models of breakdown	10
1.2.1 The model of thermal breakdown	11
1.2.2 Energetic structure of a thin film capacitor	13
1.2.3 Principles of conduction	17
1.2.4 Non-thermal models of breakdown	19
1.3 Statistical questions	27
CHAPTER II	29
2. A Class of Stochastic Models	29
2.1 Pure birth and birth-death models	30
2.2 The quasi-stationary model	32
2.2.1 The charge accumulation process	33
2.2.2 The quasi-stationary distribution	36
2.2.3 Computational aspects	41
2.3 Relevance of Extreme Value Theory	45
2.4 Experimental evidence	48

Contents - Cont'd

	Page No.
CHAPTER III	56
3. Extreme Value Theory - a Survey of Relevant Results	56
3.1 The domain of attraction problem	56
3.2 The joint distribution of extreme order statistics	68
3.3 Applications to statistical inference	71
CHAPTER IV	82
4. Finite Charge Breakdown Models	82
4.1 Domain of attraction	82
4.1.1 Zero initial charge	82
4.1.2 Quasi-stationary case	85
4.2 Regenerative case	86
4.3 Non-regenerative case	87
4.3.1 Zero initial charge	88
4.3.2 Quasi-stationary case	89
CHAPTER V	90
5. Infinite Charge Breakdown Models - Pure Birth Case	90
5.1 Domain of attraction	91
5.1.1 Exact form of the distribution function F(t)	92
5.1.2 Quadratic rates	96
5.1.3 General approach	98
5.1.4 An alternative approach	107

Contents - Cont'd

	Page No.
5.2 Normalizing constants	109
5.2.1 Power charge accumulation rates	109
5.2.2 Geometric charge accumulation rates	123
5.2.3 The direct approach in the geometric rates case	137
5.3 Inference for the pure birth model	138
5.3.1 Regenerative case	139
5.3.2 The non-regenerative case	148
CHAPTER VI	150
6. A More General Model of Breakdown. A Tauberian Theorem	150
6.1 The infinite charge model based on transient birth-death process	150
6.2 A Tauberian theorem	159
6.3 Special case	171
CHAPTER VII	174
7. Conclusions	174
7.1 Resume	174
7.2 Open problems	176
Appendix A. Basic properties of theta-functions	179
Appendix B. Euler's summation formula	183
Appendix C. On the saddle point method	186

Contents - Cont'd.

		Page No.
Appendix D.	The classification of birth and death processes	191
Appendix E.	Choice of the appropriate function $\theta(\gamma,N,\beta)$ when finding the normalizing constants in the case $\lambda_j = a(j+1)^{\alpha}$	197
Appendix F.	Eigenvalues and eigenvectors of the infinitesimal matrix A	199
Appendix G.	Transforming the contour of integration and mapping in the integral form of F(t) in the case $\lambda_j = b\lambda^j$	202
Appendix H.	On oscillatory behavior of $\phi(q)$ in the case $\lambda_j = b\lambda^j$	204
Appendix I.	Justification of the termwise passage to the limit for finding $\tilde{k}$ in the case of power charge accumulation rates.	205a
Appendix J.	Properties of the estimators $\widehat{1/\gamma}$ and $(\widehat{\ln b_N})$	205c
	References	206

ABSTRACT

In this thesis we investigate the properties of stochastic processes which are used to model a type of breakdown mechanism in thin film capacitors subjected to test under high electric fields. The insulating film in the capacitor is regarded as a collection of N cells operating independently. In every cell a stochastic breakdown mechanism of the same type is under way. It proceeds until the first of the cells actually fails and causes a breakdown in the insulator as a whole. At such a time the processes in other cells either remain unaffected or return to an initial state, depending on experimental conditions and properties of the insulating material, and the mechanism restarts. Otherwise the only effect of the breakdown is a decrease in the number of operating cells.

In this work we consider those mechanisms for which breakdown in a cell may be modelled as the explosion of a birth-death process. When analysing a given model, we are interested in the behaviour of the resulting observed process of successive breakdowns and its possible dependence on the applied voltage, insulator area and other parameters. The key tool used in investigating this observed process is Extreme Value Theory. For the case when breakdown is caused by the explosion of a pure birth process, we develop an approach based on saddle point approximation method which simplifies the problem considerably in certain situations. This approach is used in two important cases. In the sixth chapter we prove a Tauberian theorem which relates the asymptotic behaviour of a distribution function to that of its

Laplace transform. This theorem may enable one to cope with the situation when the breakdown mechanism is of a more complicated nature than those mentioned above, and particularly when the transform is not regularly varying.

When the capacitor is subjected to test under an electric field, we may observe realizations of the process of successive breakdowns under various experimental conditions. These data may be used to help identify the breakdown mechanism which is operative in a given situation (or, at least to reject those mechanisms which are not consistent with the data). After the identification has been performed, the data may also enable one to estimate the parameters of the model of interest. These problems of identification of the operating breakdown mechanism and estimation of its parameters are also considered in the present work.

CHAPTER I

1. INTRODUCTION

This thesis is devoted to the study of extreme value properties of absorption or explosion times in some stochastic processes. We consider the sys tem of a large number of similar processes, operating independently. As soon as one of the processes reaches some critical state, it dies and causes a breakdown in the system. The death of a single process affects the remaining processes in some prescribed way and they continue operating until the following breakdown occurs and so on. In this work we investigate the statistical properties of the process of successive breakdowns, the key tool of investigation being Extreme Value Theory.

The research was motivated by the physical phenomenon of dielectric breakdown in thin film insulators which we describe in some detail so that researchers in that field may examine their experimental results in the light of Extreme Value Theory.

Interest in the study of thin film insulators has been greatly stimulated in recent years because of their numerous applications in industry and technology. Insulating films play an important role in such fields as integrated circuits, copying, display devices, measurement devices etc. In all the mentioned fields the reliability of the elements involving thin films is of great importance. For example, the failure of the thin film insulator which is used as an element of memory in a computing device may lead to the change of the stored information and to losses connected with it.

A primary failure mode of today's electronic devices using insulating films is known to be dielectric breakdown. This failure mode is due to latent defects which occur randomly in the film during fabrication as well as to the internal processes taking place in the insulator when it is subjected to high fields. The physical processes are today less understood in insulators than in standard semiconductors and this is also the case for processes causing the breakdown. The explanation of the breakdown mechanism is usually a complex physical problem - difficulties arise both for theoretical and experimental reasons.

Breakdown is interpreted in terms of mechanisms such as thermal breakdown and electronic impact ionization, as well as ion-induced and other mechanisms. However, identification of the appropriate one in every concrete case still remains a difficult task. Moreover, we cannot expect that a given mechanism will be pertinent under different experimental conditions. As is pointed out in Klein (1978), it is possible that several mechanisms are operating at the same time and at a given field breakdown arises by that mechanism which is fastest; also in a given insulator different breakdown mechanisms can be operative in different temperature ranges.

The breakdowns occurring in thin film insulators at high fields (when at least one of the electrodes is a thin metal film) are of very special nature - they are typically non-shorting. After a breakdown occurs in some place, the region of the point of breakdown evaporates and the insulator returns to its normal operating conditions, except that its area becomes smaller. (See paragraph 1.1.1 for more detailed explanation). In most cases this

decrease of area can be neglected. So, we can observe the process of successive breakdowns using the same sample. We believe that the statistical analysis of the observed realization of the mentioned process may be useful in identifying the breakdown mechanism which is appropriate in a specific case (or, at least, such analysis can help to reject the mechanisms which are not appropriate). The observed data may also be used for estimating the parameters of the model which is believed to be relevant in the given situation.

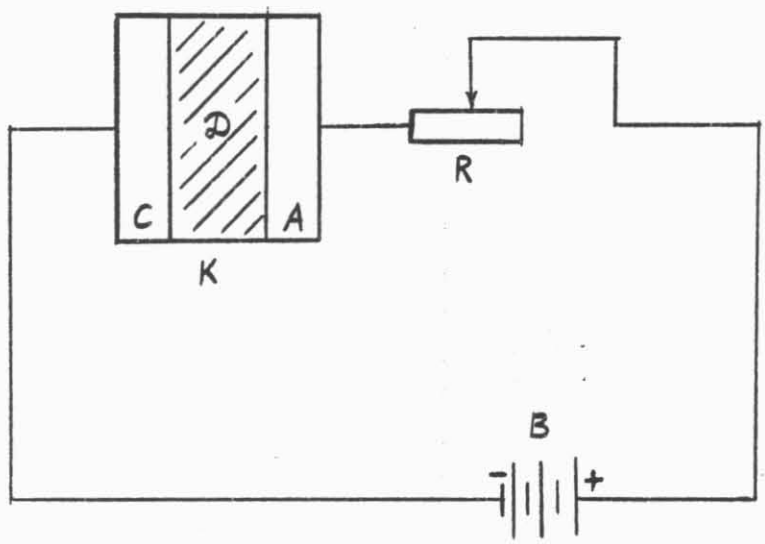
In this work we consider a number of stochastic breakdown mechanisms based on well known phenomena of impact ionization and trapping (These are considered today as the most probable factors explaining the breakdown phenomena at usual working temperatures. However, the method of investigation may be applied to other breakdown models as well). For most of the considered models we derive the statistical properties of the process of successive breakdowns and consider the problems of estimation for the parameters of the model.

1.1 The breakdown phenomenon

In this section we describe in detail the breakdown phenomena and the data which is at our disposal.

1.1.1 Description

Let us consider the following electrical circuit:



Here B is the battery, R is a resistor-regulator, K is the capacitor, C,A and D are respectively the cathode electrode material, the anode electrode material and the insulating material (thin film in our specific case).

A sample used in one of the typical experiments is shown in Fig. 1.1.1. The voltage between the electrodes (which determines the external field applied) can be regulated by means of the resistor. When the field becomes greater than some prescribed critical value, a current runaway may occur at a certain point in the insulator (like a short circuit). This leads to a local rise of temperature and evaporation of the insulating material and electrode materials in the region of the current runaway (and to self liquidation of this point). This phenomenon is called a non-shorting

(or self-healing) breakdown. As a result of such a breakdown a hole remains in the capacitor and after the field is removed we have a capacitor which is equivalent to a new one.

When a new capacitor is subjected to test by field, breakdown may occur at relatively low voltages. This happens because of the presence of so called "weak spots" - defects in the structure of the insulating film which occur in the fabrication process. But after these spots are removed by non-shorting breakdowns the process of breakdowns usually develops in such a way that breakdowns have no obvious tendency to form clusters both in the area and time. The rate of breakdowns tends to increase when increasing the applied voltage.

The described process of breakdowns is sometimes interrupted by the occurrence of so called propagating breakdowns. Such breakdowns are accompanied by a series of single hole breakdowns occuring at adjacent sites or even by the appearance of an arc between the pit of the initial single hole breakdown and the metal of the counter-electrode. Though propagating breakdown is non-shorting, it causes evaporation over a large area - and this loss cannot be neglected after the capacitor returns to its normal operating conditions. (See also O'Dwyer (1973, p. 280)).

Since propagating breakdowns may be prevented by the properly chosen protective resistor which is connected in series to the tested capacitor to prevent high discharge currents, we exclude them from further consideration.

There is one more kind of breakdown, called shorting. After such a breakdown occurs, there remains a permanent conducting channel in the insulator which destroys the capacitor as a whole.

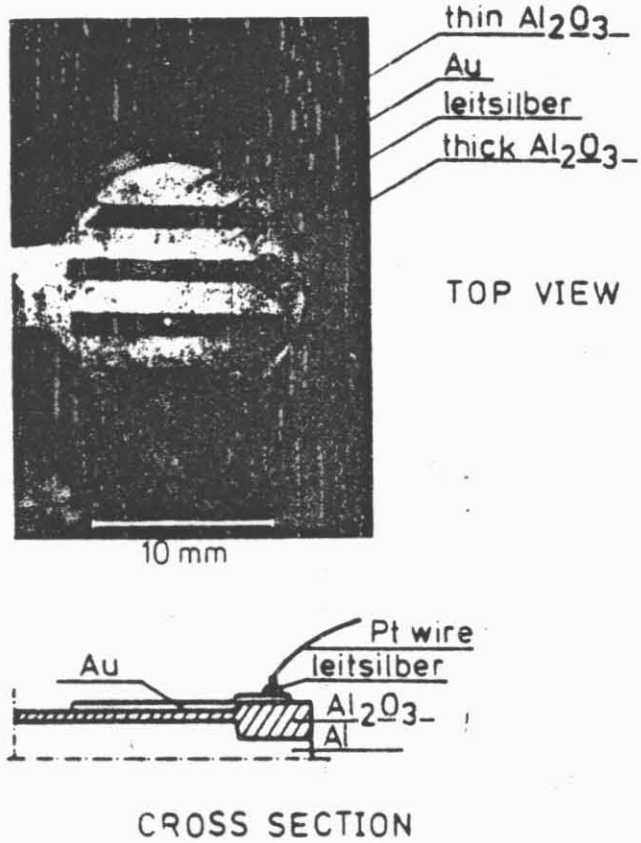


Fig. 1.1.1 A typical specimen (From de Wit et al (1976)).

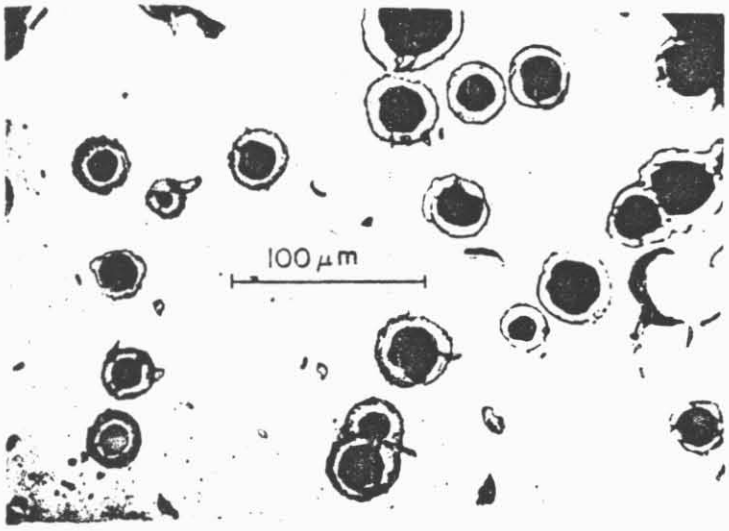


Fig. 1.1.2 Breakdowns in the capacitor with thin film insulating material. (From Klein (1971)).

In Fig. 1.1.2 we can see the part of a capacitor subjected to a high field after the occurrence of several breakdowns.

1.1.2 Data available

The data available are several sets of successive times between breakdowns for various insulating materials, for different voltages applied and for different insulator thicknesses. Also, several experimental graphs, representing the dependence of the critical breakdown field on the film thickness, the dependence of the rate of breakdown on the voltage, and the dependence of the rate on the film thickness are available. At our disposal also are data reflecting the percentage of insulator surface destroyed against time (for different voltages applied). Numerous experimental data are found in the papers devoted to the study of breakdown phenomena in such widely used insulating materials as silicon dioxide, oxide of aluminium, silicon nitride etc. (See the References).

Typical set of successive interbreakdown times is represented in Fig. 1.1.3.

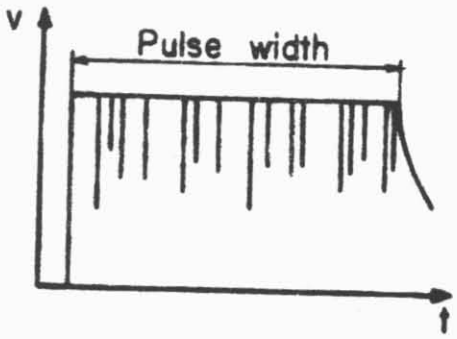


Fig. 1.1.3. A Typical Set of Successive Interbreakdown Times.

Most of experiments with thin film capacitors are performed by subjecting the sample to constant voltage, as described before. Such experiments are called step voltage tests. However, of great importance is information obtained by ramp voltage tests in which a sample is subjected to voltage increasing linearly in time. After a breakdown occurs and voltage drops to zero, it keeps increasing at a constant rate until the following breakdown and so on.

Another possibility is to subject the sample to stressing by constant current. However, shorting breakdowns typically occur in such tests and therefore they are rarely used.

Data, obtained in ramp voltage tests and constant current tests are also available.

1.2 Some physical models of breakdown

In this section we review shortly the main principles (and connected models) which are used today to explain the breakdown phenomena in thin film insulating materials. Two classes of mechanisms which are generally used are the thermal and electrical mechanisms. First of all we explain the principles of the thermal breakdown in dielectrics. To describe the electrical models of breakdown we need first to represent (at least qualitatively) the energetic structure of the thin film capacitors and the principles of conduction in insulating films. We shall do it in two separate sections.

1.2.1 The model of thermal breakdown

A thermal breakdown in the thin film capacitor is caused by the instability which arises when the Joule heat, generated by the current flow cannot be conducted (quickly enough) to the surrounding space. When such a situation is present, an unbalanced rise in temperature leads to irreversible changes in the insulating material and breakdown in one of the hottest points.

If the only significant process of the energy loss in the capacitor is thermal conduction to surroundings (another possible process is, for example, light emission), the insulator is given by region D in the three-dimensional space and field is applied through the electrodes of the capacitor at time t=0, the temperature $T(x,y,z;t)$ in the point $(x,y,z) \in D$ at time t satisfies the partial differential equation of the second order (See O'Dwyer (1973, p. 10)).

$$C \frac{\partial T}{\partial t} = \mathrm{div}(\kappa \ \mathrm{grad}\ T) + \sigma F^2 \quad , \tag{1.2.1}$$

where C is the specific heat per unit volume, κ and σ are the thermal and electrical conductivity respectively, $F(x,y,z)$ is the field and grad T is the space gradient of the temperature.

We can see that (1.2.1) is just an equation of energetic balance at the point (x,y,z) : the rate of gain in energy at the point is equal to the rate of gain due to thermal interchange with the neighboring points plus the rate of gain from the field. If we are given the appropriate initial and boundary conditions, the equation (1.2.1) may be solved (at least numerically)

resulting in the temperature as a function of time and coordinates. To solve (1.2.1) analytically is possible only for the simplest boundary conditions since σ usually depends strongly on temperature (it may depend on field as well) and κ is also temperature dependent. However, we do not need the full solution. As it is pointed out by O'Dwyer (1973, p. 10), failure of the dielectric depends on the temperature of its hottest part, so only the numerical solution in this part is needed. The position of this part can sometimes be read just from initial and boundary conditions. Several solutions of (1.2.1) for the case that constant voltage is applied at time t=0 are shown schematically in Fig. 1.2.1. What can be said in general about the (hottest part) solution in this case is that there exists a critical voltage V* for which the temperature of the hottest part asymptotically approaches some critical temperature T*. Then if the applied voltage is $V > V^*$, the temperature reaches any level in finite time; otherwise it asymptotically approaches a level $T < T^*$. If we know what temperature causes breakdown, it is possible then to calculate for which voltages the breakdown may occur and what will be the life time of the insulator (See O'Dwyer (1973, Section 1.3 and Chapter 6)).

Finally, let us note that almost all insulators should undergo thermal breakdown at a sufficiently high temperature, since the electrical conductivity is usually an exponentially increasing function of the temperature and the thermal conductivity is usually a decreasing function of the temperature (See O'Dwyer (1973, Section 9.2)).

However, it often occurs that the applied field cannot raise the temperature significantly (it certainly remains below the temperature,

causing thermal breakdown) and nevertheless a breakdown occurs in the insulating material. In these situations it is generally one of the electrical breakdown mechanisms that is operative.

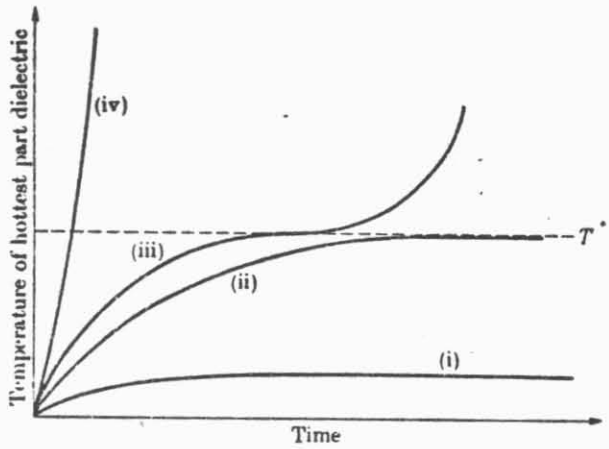


Fig. 1, 2, 1, Solution of the equation (1, 2, 1) in the hottest region for the cases $V < V^*$ (i), $V = V^*$ (ii), $V > V^*$ (iii) and $V >> V^*$ (iv). (From O'Dwyer (1973)).

1.2.2 Energetic structure of thin film capacitor

As it is known from the band theory of solids (See Rose, Sheppard and Wulff (1966, Chapter 1)), the electrons in a crystal may have only prescribed energy levels which form the valence band and the conduction band (high energy band). To move free in the crystal, the electron must have an energy which is greater than the "bottom" of the conduction band. The two bands may overlap (like in the case of metals) and then the free charge carriers (which ensure current as soon as voltage is applied) are

always present. In the case of dielectrics,however, the conduction band is empty and there exists a gap E_g between the "top" of the valence band and the "bottom" of the conduction band. The crystal is usually called a dielectric if $E_g > 2ev$ (electron-volts) and a semiconductor if $E_g > 2ev$. The interband energy levels form a so called forbidden band.

This energetic picture is appropriate only when speaking about pure and defect-free non-metals. When inpurities or mechanical defects are present, additional energy levels may be present in what is normally the forbidden band. These levels are localized in the small regions, called "trapping centers" (or traps). When these additional (they are also called extrinsic) levels are close to the "top" of the valence band, they may be easily occupied by (thermally or otherwise) excited electrons. The positive charge which remains then in the valence band is called a hole. The movement of the hole in the valence band (which is caused by movement of electrons in this band) is completely analogous to the movement of electrons in the conduction band. The movement of holes proceeds in accordance with a diffusion process (assisted by an external field when it is present).

It is well known, that the electrons in the conduction band of metals are distributed (in respect to energy) in accordance with Fermi's distribution law. The energy of the highest filled state in the conduction band at OK (Kelvin's scale) is called a Fermi Level (and denoted by E_F). This distribution only slightly depends on temperature and only a relatively small proportion of the electrons occupy the levels higher than E_F at high temperature (the last fact explains why the metals remain intact at high temperatures).

When constant field F is applied to the thin film capacitor, its energetic structure may be understood from Fig. 1.2.2. The Fermi Levels of the cathode and anode are denoted by E_{F1} and E_{F2} respectively. E_V and E_C are the "top" of the valence band and the "bottom" of the conduction band. E_T is one of the trap levels. The slope is proportional to the applied field F. It is clear from the plot, what must be the minimal energy of an electron in a given point of the insulator which enables its free movement. We see that an electron must overcome the potential barrier ΔE before it enters the insulator. Similarly, it must overcome the barrier ΔE_T before it is able to get free from the trap. An electron may overcome the potential barrier in two ways:

a) it can gain additional energy greater or equal to the height of the barrier. Then thermal emission or photoemission from the cathode into the insulator occurs dependent on the source of energy.

b) it can penetrate the barrier by tunneling (which is a purely quantomechanical effect).

Actually, when the electron enters the insulator it leads to the appearance of the local "image" positive charge which distorts the field in the relevant region and lowers the potential barrier. A similar effect takes place when electron leaves the trap. This phenomena is called Schottky's effect (See O'Dwyer (1973, Chapter 3)). If this effect is taken into account, the energetic diagram of the capacitor corresponds to the dotted line in Fig. 1.2.2. We can see, that the potential barrier at the cathode is reduced by an amount ΔE_S (which is known to be proportional

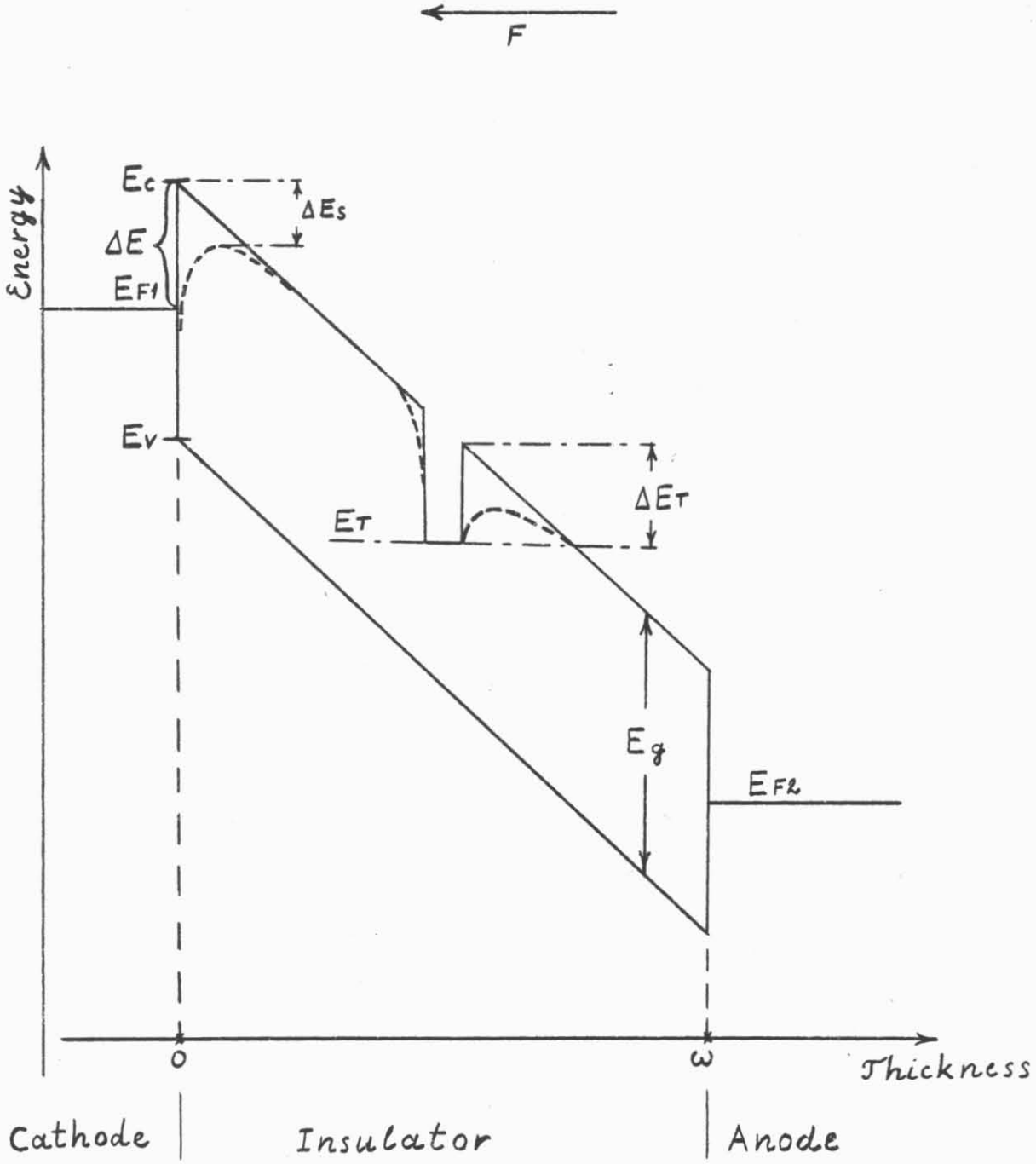


Fig. 1.2.2 Energetic structure of thin film capacitor, subjected to constant field.

to the square root of the applied field). The lowering of the barrier when leaving the trap is known to be equal to $2 \times \Delta E_s$.

Analogous considerations can be made concerning the injection of holes at the anode, their movement and trapping. In the following section we breefly describe the mechanisms of conduction in thin film capacitors.

1.2.3 Principles of conduction

In this paragraph we describe the process of current conduction in thin film insulators. We shall assume that the main charge carriers are electrons. The situations in which both electrons and holes play a significant role in charge conduction are much more complicated and rare.

Consider a capacitor which is subjected to high field F. The electrons which produce the current in the insulator are injected into its conduction band mainly by two processes - Fowler-Nordheim (tunelling) emission and Schottky (thermal) emission. There is one more process which may be significant when a high degree of impurity is present and field is not too high - the so called hopping process (See O'Dwyer (1973, p. 71)). This process is usually neglected when studying the breakdown properties of thin film insulators. If the temperature of the cathode is T, the injected current densities are (Schottky barrier lowering is taken into account) :

(1.2.2) $$j_{thermal} = AT^2 \exp\{- \frac{\Delta E - B\sqrt{F}}{kT}\}$$

(1.2.3) $$j_{tunel} = CF^2 \exp\{- \frac{D}{F}\}$$

where A, B, C, D are constants depending on physical properties of the cathode and insulator (their exact expressions may be found in Klein (1972, p. 617)), k is the Boltzmann constant and ΔE is the height of the potential barrier at the cathode when no field is present (See Fig. 1.2.2). Usually one of these injection mechanisms is dominant and the second is neglected. The thermal emission is dominant at high temperatures and not too high fields applied. The tunelling emission usually is dominant when high fields are applied even at device - operating temperatures. Note that the appropriate current (given by (1.2.3)) does not depend on temperature.

However, the current in the insulator is not determined by injected current only and the following factors should also be considered.

a) Density of trapping centers and the depth of the potential barrier, associated with them. Trapped electrons weaken the field in the cathode region and therefore decrease the current injection. Trapped holes act analogously and enhance the cathode field.

b) The occurrence of ionization affects the current by producing additional charge carriers in the insulator. Drift (diffusion) of the products of ionization and recombination processes have influence on the current too.

c) The scattering effect of the crystallic lattice and mobility of charge carriers in the insulator.

When a particular case is considered, we determine the current injection type, decide what factors must be taken into consideration and then solve the appropriate equations obtaining the quantitative characteristics of the system. For example, characteristics of breakdown may be obtained by analysing the dependence of current density on time, field applied, thickness of the insulator and other relevant factors.

1.2.4 Non-thermal models of breakdown

In this paragraph we review some typical non-thermal models which are widely used to explain the breakdown phenomena in various substances. This review by no means claims to be complete.

We begin from the models in which the phenomena of impact ionization plays a major role. A certain type of carriers (usually electrons) are injected from an appropriate electrode and ionize the atoms of the insulating material by impact. The second type of carriers do not take a significant part in current conduction because their mobility is usually low (in fact, they are deep trapped). However, they affect the current by distorting the field in the region ofthe injecting electrode. (This situation is thought to be relevant in the case of wide interband gap insulators. (See Klein (1978, p. 225)). For narrow interband gap dielectrics and semiconductors both types of carriers may play a significant direct role in current injection and ionization processes). Under these circumstances different mechanisms of instability (and breakdown) apply. In the case when electrons are the dominant charge carriers breakdown is caused by positive feedback effect as described in Klein and Solomon (1977). At high fields

electronic impact ionization occurs which leaves positive holes behind drifting slowly towards the cathode. The holes enhance the field at the cathode, increase the charge carrier emission and lead to an increase in impact ionization. The effect of this process can be opposed by processes such as removal of holes from the insulator by drift and recombination. Drift and recombination can balance the effect of impact ionization below a critical average field $\overline{F}^*$, but above $\overline{F}^*$ current runaway arises after times t^* which decrease quasi-exponentially when increasing the average field $\overline{F}$.

The energetic diagram of the described phenomena is shown in Fig. 1.2.3. Note, that an electron can ionize only if its energy is greater than the bottom of conductivity zone by at least E_g.

Several models of breakdown are based on this scheme dependent on the hole removal mechanism. The first analysis is due to O'Dwyer (1969), who assumed that the dominant hole removal mechanism is drift (such assumption leads to so called ionization-drift (ID) model). In this work the author presents only a numerical method for finding $\overline{F}^*$ and t^*. Closed (though approximate) expressions for these values were given by Kashat and Klein (1977). The approximate nature of their approach is caused by the assumption that the trapped positive charge is distributed uniformly in the insulator. In fact this space charge is distributed exponentially in respect to thickness (See Fig. 1.2.4).

The ID models are subdivided into two classes (dependent of the mobility of holes and ionization rate per unit length in the given situation) - the Small Ionization and Large Ionization ID models. The difference between these models is that in the first the injected electrons ionize by impact

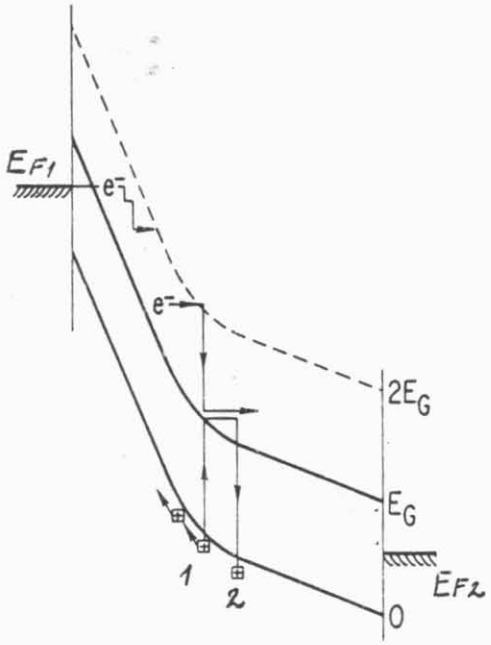


Fig. 1.2.3 An energetic diagram of events which occur in the beginning stages of breakdown in silicon dioxide. 1. corresponds to a neutral atom which turns a hole after collision occurs; 2. is a hole which turns to neutral atom after recombination occurs. (From DiStefano and Shatzkes (1977)).

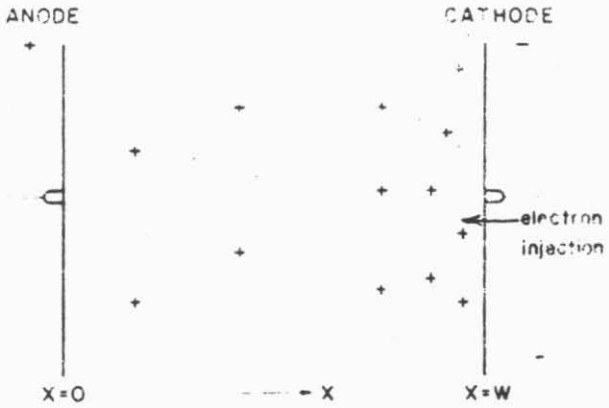


Fig. 1.2.4 Schematic cross section of insulator flanked by electrodes. (From Klein and Solomon (1976)).

only occasionally while in the second one they produce small avalanches (clusters of trapped holes appearing as a result of ionization produced by a single injected high energy electron). The exact criteria for application of the first or second ID model and the expressions for the critical average field $\overline{F}^*$ and time to breakdown t^* may be found in the work of Kashat and Klein (1977).

In the case that hole mobility is extremely small, hole removal proceeds mainly by recombination and the IR (ionization-recombination) model is operative. This model was treated first by DiStefano and Shatzkes (1975) to explain the breakdown phenomena in silicon dioxide thin films. In their approach current was emitted at the cathode by Fowler-Nordheim tunelling (a boundary condition), and stochastic Poisson model of ionization was used. After finding the equilibrium state space charge distribution and the form of energetic diagram, they calculate the critical voltage at which instability occurs. The questions concerning time to runaway were not, however, discussed.

The closed expressions for $\overline{F}^*$ and t^* (IR model) were obtained by Klein and Solomon (1976). To do that, the authors formulated and analysed the set of four basic relations

(1.2.4) $$j = AF_c^2 \exp(-\frac{B}{F_c})$$

(1.2.5) $$\frac{\partial F}{\partial x} = \frac{qp}{\varepsilon_o \varepsilon}$$

(1.2.6) $$\frac{\partial p}{\partial t} = \frac{j}{q} (\alpha - \sigma p)$$

$$\alpha = \alpha_o \exp(-\frac{H}{F}) \tag{1.2.7}$$

The assumption that the insulator is planar has been made (See Fig. 1.2.4). Equation (1.2.4) shows that Fowler-Nordheim emission is supposed to be the dominant current injection mechanism (F_c is the cathode field). The Poisson equation (1.2.5) relates the spatial distortion of field to the trapped positive charge density p. Here q is the charge of the electron, ε_o is the absolute permittivity of the free space and ε - the relative permittivity of the insulating material. (1.2.6) is the rate equation which means that the rate of positive charge accumulation in a given point is equal to rate of its generation minus the rate of recombination in this point. σ is the recombination cross section and α is the ionization rate per unit length; its dependence of F is postulated by equation (1.2.7) (Motivation of such assumption may be found in O'Dwyer (1973, p. 220)). A,B and H are appropriate constants. We note that similar approach was used to obtain the characteristics of the ID model in the above mentioned works on this subject.

By reviewing the known properties of silicon dioxide, Solomon (1977) comes to the conclusion that IR is a most probable mechanism of breakdown in this dielectric. It is also thought to be operative in some alkali halides.

The mentioned models enable one to obtain (analytically or numerically) the main deterministic characteristics of breakdown - $\overline{F}^*$ and t^*. However, the common experimental situation is that breakdowns show stochastic behaviour in time and often occur at fields which are lower than the

calculated critical field. This kind of behaviour is usually ascribed to fluctuations caused by discreteness of the electronic charge. The effect of fluctuations was studied in the "succession of avalanches" model, proposed by Klein (1972) for investigating the Large Ionization ID case. In this model electrons, injected from the cathode, produce avalanches by impact ionization, leaving clusters of low mobility holes in the insulator. The vast majority of the avalanche charges drift out of the insulator without further effect. There is however a chance that another electron is injected into the cluster before it leaves the insulator. A second avalanche arises which locally enhances the cathode field and greatly increases the probability of injection of an additional electron into the cluster and so on. When a series of several avalanches occurs at one spot, a current runaway (and breakdown) is observed. In Kadary and Klein (1980) this model is shown to explain the breakdown in anodically grown tantalum pentoxide films.

A purely stochastic model of breakdown was proposed by Solomon, Klein and Albert (1976) to explain the fluctuations, arising when thin insulator is subject to test by constant and linearly increasing field. The authors postulate that under a constant field breakdowns form a Poisson process with rate which depends exponentially on the applied field. The experimental results for constant and linearly rising fields were in good agreement with those predicted by the model for insulators such as aluminium and hafnium oxides. The relation between mean times to breakdown in both types of tests, predicted by the model, was also supported by observations. A similar model (in which the linear form of dependence

between area of insulator and rate of breakdown under a constant field was an additional assumption) was used to explain the breakdown in aluminium oxide by de Wit, Wijenberg and Crevecœur (1976).

Another important class of models arise when breakdown phenomena is related directly to the ionic movement in the insulator. In one of such models, proposed by Ridley (1975) to explain breakdown in silicon dioxide films, an existence of small protrusions in the cathode material is postulated. The magnitude of such protrusions is assumed to be of order 100Å. Since the magnitude of protrusions is considerable compared with film thickness (it is usually of order 1000Å), the field is much stronger in the region of the protrusion. The Fowler-Nordheim injection produces then a high current density (and, consequently, high degree of ionization) in the relevant filamentary region.

The ionization process is accelerated by the rise of temperature in this region (which is caused by high current density). The produced electrons are swept out while the positive ions drift slowly to the cathode, enhance the field there and in this way lead to a positive feedback process. A breakdown then occurs because of thermal instability in the appropriate region (provided that the applied average field is greater than some critical value $\overline{F}^*$). The closed expression for $\overline{F}^*$ was not, however, given and this model was criticized by Klein (1978) on the basis of the fact that this ion-induced model hardly predicts the observable dependence of $\overline{F}^*$ on temperature and lack of significant dependence of $\overline{F}^*$ on electrode material in silicon dioxide. (See Klein (1978, p. 225)).

There are also non-thermal variants of ion-induced breakdown mechanism. In such models field is enhanced because of positive ions which form small protrusions when undergoing recombination at the cathode and result in a decrease of the insulator thickness in the protrusion region. A positive feedback occurs which leads to breakdown. Such a mechanism is thought to be probable in ultrathin (of order 100Å) insulating films.

In the models considered before the assumption is that insulators are subject to a constant voltage. E. Harari (1977) performed a series of tests in which ultrathin (SiO_2) insulators were stressed by constant current (and voltage is merely adjusted to meet this condition). The author asserts that defects appear in the insulating layer as soon as field is initially applied. These defects act as deep trapping centers for electrons, which are disposed mainly in the cathode region. The electrons trapped in this region decrease the cathode field, so voltage must be increased to maintain the current. The process of defect creation by injected current proceeds at a uniform rate and so the applied average field increases. A breakdown occurs as soon as it becomes large enough to break the molecular bonds in the insulating material. The author does not attempt to construct an appropriate mathematical model - he only points out that this qualitative model is supported by the observations. He also indicates that there is no evidence of impact ionization in this case (such evidence is usually provided by luminescence caused by recombination).

Finally, we note that the trapping centers for electrons play an important role in most recent theories explaining the behaviour of the

dielectric strength of aluminium dioxide films subjected to step and ramped voltage tests. These theories are motivated by the experimental evidence that after a constant voltage is applied, the current density at first (usually sharpely) decreases and then increases in time. The initial decrease of current density occurs because the injected electrons fill in the traps, decrease the cathode field, and subsequently the injected current; the further increase of current density is due to impact ionization and hole trapping processes which proceed in parallel (N. Klein, 1981, private communication).

More information on the breakdown mechanisms may be found in Klein (1978) and O'Dwyer (1973). An extensive list of references concerning breakdown phenomena in thin film insulators is given in the bibliographical survey by Agarwal (1974).

1.3 Statistical questions

In this section we consider the role of statistician in analysing breakdown phenomena.

Suppose that we want to know whether a specific mechanism causes breakdown in a given dielectric. The first statistical question arises immediately: how to design the experiment. To what conditions must the capacitor be subjected in order to make it clear of weak spots? What is the appropriate criteria to distinguish between the initial "weak spot" breakdowns and those reflecting the intrinsic properties of the given insulator? Should we apply a step voltage test, a ramped voltage test or a stressing by a constant current to the capacitor after it is clear

of weak spots? Should we remove the field for a constant time after a breakdown occurs?

After the capacitor is cleared and an appropriate test is applied, we obtain a realization of the process of breakdowns. The second statistical question is: what process of breakdowns is predicted by the model of interest and does the observed realization contradict significantly this prediction? What changes in the process of breakdowns are predicted by the model when varying its parameters and do the observed processes (corresponding to appropriately changed experimental conditions) support the prediction?

Suppose we find that the model is supported by data in a certain range of experimental conditions. The third statistical question is: how to use the data to obtain estimators (and perhaps confidence intervals) for the parameters of the model? There are cases that some of the parameters may be estimated using a purely physical argument and they should be compared with statistical estimators.

CHAPTER II

2. A CLASS OF STOCHASTIC MODELS

In this chapter we describe several stochastic models of the breakdown phenomena in thin film insulators. Let us suppose that the surface of the insulator is divided into N equal cells. Under a sufficiently high electric field the current in each cell may lead to the ionization of atoms of the insulating material, creating a positive charge which is trapped in the insulator. This charge enhances the field and hence the current in the cell, and so increases the chance of further ionization. Charge accumulation processes procced in all the cells simultaneously and as soon as the charge accumulated in a cell reaches some critical value this cell explodes causing breakdown in the insulator as a whole. Our basic assumption is that the charge accumulation process in a given cell has no influence on the charge accumulation processes in other cells. A breakdown in the insulator reduces the external field to zero and some time elapses before it reaches its nominal value. It is known that this has some effect on the charge accumulated in other cells (it tends to decrease because of the internal coulomb forces acting in each cell). In some cases this effect can be neglected (i.e. explosion of cell has no influence on other cells). These cases lead to <u>non-regenerative</u> models. The <u>regenerative</u> models are connected with the cases that after a breakdown occurs in some cell, all the other cells lose their accumulated charge and the charge accumulation processes start again in non-exploded cells. For these models the times of successive breakdowns form a renewal process.

In finite models the critical value of the charge accumulated in a cell is finite. In infinite models a cell explodes when the charge, accumulated in this cell reaches infinity.

In subsequent sections we describe in detail several stochastic models of the breakdown phenomena.

2.1 Pure birth and birth-death models

In this class of models the initial charge at time t=0 is assumed to be zero in every cell. (This assumption is not so restricting as it seems; actually "zero" is a prescribed physical level). The charge accumulation in a cell proceeds in accordance with a birth-death process with "birth" rates $\{\lambda_o, \lambda_1, \ldots\}$ and "death" rates $\{\mu_1, \mu_2, \ldots\}$. In the case that all the "death" rates are equal to zero, the charge accumulation process in a cell is a pure birth process.
We consider the following models:

a) Finite models. For this class of models the charge accumulated in a cell may take on the values $\{0,1,\ldots,M-1\}$. We say that a cell is in state $j \in \{0,1,\ldots,M-1\}$ if the accumulated charge in this cell is equal to j. The insulator consists of N cells, operating independently, and as soon as the charge accumulated in one of the cells reaches M the cell explodes and a breakdown is observed. We consider also the possibility that a breakdown in a cell occurs D units of time after the charge accumulated in this cell reaches M(D is a non-negative constant).

In the regenerative case all the cells are in state 0 at t=0 and the

charge accumulation processes start simultaneously in all N cells of the insulator. After one of the cells reaches the state M and breakdown occurs, this cell disappears, (N-1) remaining cells return to the state 0 and the process of charge accumulation starts again. Since the number of the cells N is expected to be large (N is of order 10^{10}) the effect of decreasing the number of cells as a result of breakdown is neglected and we may suppose that the N cells restart after a breakdown occurs. In this model the process of successive breakdowns is a renewal process. If $\xi_i (i = 1,2,\ldots,N)$ is the time to explosion of the $i^{\underline{th}}$ cell (when considering that cell in isolation and supposing that it is in state 0 at t=0) then $\xi_1,\xi_2,\ldots,\xi_N$ are independent and identically distributed random variables and the interbreakdown time T (the interevent time of the renewal process) is

$$T = \min(\xi_1,\xi_2,\ldots,\xi_N) \tag{2.1.1}$$

The random variable ξ_i is distributed as the first passage time from the state 0 to M of the birth - death process with "birth" and "death" rates $\{\lambda_i,\ i = 0,1,\ldots,M-1\}$ and $\{\mu_i,\ i = 1,2,\ldots,M-1\}$ respectively.

In the <u>non-regenerative</u> case all the cells are in state 0 at t=0, the process of charge accumulation starts simultaneously in all N cells and a cell explodes as soon as it reaches the state M, as in the previous case. However, an explosion of a cell does not affect the charge accumulation processes in other cells. Therefore the process of breakdowns in this case is a sequence $\{\xi_{(1)},\xi_{(2)},\ldots\}$, where $\xi_{(i)}$ is the $i^{\underline{th}}$ term in the ordered sequence of times to breakdown in the individual cells.

b) Infinite models. For this class of models the charge, accumulated in a given cell takes on the values $\{0,1,\ldots\}$. The charge accumulation process in a cell is transient and reaches infinity in finite time with probability one. This happens when $\{\lambda_i\}$, the rates of ionization in a cell are large and cannot be balanced by the rates $\{\mu_i\}$ of the opposing process (which may be recombination or drift). In certain cases the effect of this opposing process may be neglected altogether and then the process of charge accumulation in a cell becomes a pure birth process.

In the regenerative case all the cells are in state 0 at t=0, the processes of charge accumulation starts simultaneously in all N cells and all the cells return to the state 0 immediately after a breakdown.

In the non-regenerative case unlike the regenerative case, breakdown has no influence on the charge accumulation processes in the non-exploded cells.

If we suspect that the process of breakdowns corresponds to a non-regenerative model, the experiment should be carried out in a special manner. At first we apply a critical field to the insulator in order to destroy the "weak spots". Then we should drop the field altogether for some time (in order to remove the trapped charge). Then we apply a fixed field again and register the moments of the successive breakdowns.

2.2 The quasi-stationary model

In this paragraph we consider the following model of the breakdown phenomena. Charge accumulation in each of N cells proceeds in accordance with a birth-death process. A cell explodes (and causes a breakdown) D

units of time after it reaches a critical state M. Explosion of a cell does not affect charge accumulation processes in other cells. Let us suppose that D=0 i.e. that the cell explodes immediately after the charge reaches the level M. The final conclusions concerning this model (See Section 4.1) can be easily re-formulated for positive values of D.

2.2.1 The charge accumulation process

We consider the process of charge accumulation in a single cell. Suppose that at time t=0 the cell is in state i ($0 \leq i < M$). Denote by $P_{ij}(t)$ the probability that the cell is in the state j at time t ($0 \leq j \leq M$). The charge accumulation process is a birth-death process with a finite number of states (0,1,2,...,M), M being an absorbing state (as soon as a cell reaches M, it explodes and remains in this state forever). From the general theory of Markov chains (See Cox and Miller (1965)) it is known that the matrix of transition probabilities $\{P_{ij}(t)\}$, ($0 \leq i,j \leq M-1$) satisfies the equations

$$P'(t) = AP(t) \quad \text{(backward equations)} \tag{2.2.1}$$

$$P'(t) = P(t)A \quad \text{(forward equations)} \tag{2.2.2}$$

with the initial conditions

$$P(0) = I \tag{2.2.3}$$

and has the following properties:

(2.2.4) $$P_{ij}(t) \geq 0$$

(2.2.5) $$\sum_{j=o}^{M-1} P_{ij}(t) < 1 \quad \text{for every } i$$

(2.2.6) $$P(t+s) = P(t)P(s)$$ (Chapman-Kolmogoroff conditions are semigroup property of P(t) as a matrix operator).

The matrix A (which is called in the literature the "infinitesimal generator" has the form

$$(2.2.7)\quad A = \begin{bmatrix} -\lambda_o & \lambda_o & 0 & 0 & 0 & 0 & . & . & . & 0 \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & 0 & 0 & . & . & . & 0 \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 & 0 & . & . & . & 0 \\ . & & & & & & & & & . \\ . & & & & & & & & & . \\ . & & & & & & & & & . \\ 0 & 0 & . & . & . & . & 0 & \mu_{M-2} & -(\lambda_{M-2}+\mu_{M-2}) & \lambda_{N-2} \\ 0 & 0 & . & . & . & . & 0 & 0 & \mu_{M-1} & -(\lambda_{M-1}+\mu_{M-1}) \end{bmatrix}$$

All the rates λ_i, μ_i involved in A are supposed to be strictly positive. The solution of the above equations (2.2.1,2) is unique (since the number of states is finite) and having solved it the probabilities of absorption at time t ($P_{iM}(t)$) can be found from

$$\sum_{j=o}^{M} P_{ij}(t) = 1 .$$

We know that as $t \to \infty$ the probabilities $P_{ij}(t)$ converge to the limiting vector which ascribes probability 1 to state M and zero to the other states. However, if we condition on a cell being "alive", i.e. not yet being absorbed in state M, there may be some sense in which we may regard the cell as having reached a steady state.

Let $r_{i,j}(t)$ be the probability that, having initially been in state i, a cell is in state j at time t given that it has not yet reached absorption. The probability, $H(\Delta t)$, that it will be absorbed in $(t, t+\Delta t]$ given that it is still "alive" at time t is clearly given by

$$H(\Delta t) = r_{i,M-1}(t) \cdot \lambda_{M-1}\Delta t + o(\Delta t). \tag{2.2.8}$$

If $r_j = r_{i,j}(t)$ is both independent of i and t, then the vector $(r_o, r_1, \ldots, r_{M-1})$ would correspond to the stationary distribution of the process conditioned on non-absorption. The last equation would then lead to a very simple model of the breakdown phenomena in the system of cells-namely to that of a Poisson process.

In the following we refer to results which show that the $r_{i,j}(t)$ do have limits as $t \to \infty$, the so called quasi-stationary distribution. The relation $r_{i,j}(t) = r_j$ for all t will hold if the initial distribution is this quasi-stationary distribution. On the other hand, by comparing the rates at which the $r_{i,j}(t)$ converge to r_j with the rate at which the breakdowns occur in the system, it may be reasonable to assume that this quasi-stationary regime is attained after an initial period of observation. In other words, after this initial period, at the time of a breakdown the remaining cells are distributed in the states 0 to M-1 according to the distribution $(r_o, r_1, \ldots, r_{M-1})$. From this property the Poisson process of breakdowns will follow.

2.2.2 The quasi-stationary distribution

Let us consider the process of charge accumulation in a single cell. Denote by $r_{i,j}(t)$ the probability that the cell is in state j at time t (the initial state was i) given that the cell is not in the absorbing state M at time t. The conditional probabilities $r_{i,j}(t)$ are related to the transition probabilities $P_{ij}(t)$ by the equations

$$(2.2.9) \qquad r_{i,j}(t) = \frac{P_{ij}(t)}{1-P_{iM}(t)} = \frac{P_{ij}(t)}{P_{io}(t)+P_{i1}(t)+\ldots+P_{iM-1}(t)}, \qquad 0 \le i,j < M.$$

We ask now the following question: is it true that the $r_{i,j}(t)$ approach (as $t \to \infty$) a limit depending on the final state j only? These limits, if they exist, might then be identified with the proportions $r_o, r_1, \ldots r_{M-1}$ of the cells finding themselves in the states $\{0,1,\ldots,M-1\}$ respectively

at any moment of time after the steady state have been reached. It is known (See Darroch and Seneta (1965) and Keilson (1979, p. 90)) that the answer to this question is affirmative for the general Markov case with an absorbing set (instead of an absorbing state M). The quasi-stationary probabilistic vector $\{r_o, r_1, \ldots\}$ is known to be a left eigenvector of the infinitesimal matrix (reduced to the set of non-absorbing states) corresponding to its maximal eigenvalue. The modulus of this eigenvalue can be shown to be equal to the principal decay rate for the charge accumulation process on the set of non-absorbing states given that the initial distribution on this set is a quasi-stationary distribution $\{r_o, r_1, \ldots\}$. In other words, the modulus of this eigenvalue times Δt gives the main term of the proportion of the "alive" cells which will die in time Δt.

In our specific case the process of interest is a birth-death process with absorbing state M. The problem of finding the maximal eigenvalue of the infinitesimal matrix A and the corresponding left eigenvector (the quasi-stationary distribution) is largely simplified by using the set of potentials $\pi_i (i = 0,1,\ldots,M-1)$ attached to the non-absorbing states and a set of polynomials $Q_o(x), Q_1(x), \ldots, Q_{M-1}(x)$ defined by means of A.

The potentials $\{\pi_o, \pi_1, \ldots, \pi_{M-1}\}$ are defined by

$$(2.2.10) \qquad \pi_o = 1, \ \pi_n = \frac{\lambda_o \ \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \ \mu_2 \cdots \mu_n} \quad \text{for} \quad 1 \leq n \leq M-1.$$

and the polynomials $Q_o(x), Q_1(x), \ldots, Q_{M-1}(x)$ - by means of the relation (See Karlin and McGregor (1957a))

$$(2.2.11) \qquad AQ = -xQ$$

with $Q_o(x) \equiv 1$. The explicit form of the relation (2.2.11) is

$$
\begin{aligned}
& Q_o(x) = 1 \\
& -xQ_o(x) = -\lambda_o Q_o(x) + \lambda_o Q_1(x) \\
(2.2.11a) \qquad & -x\, Q_1(x) = \mu_1 Q_o(x) - (\lambda_1+\mu_1) Q_1(x) + \lambda_1 Q_2(x) \\
& \vdots \\
& -xQ_{M-2}(x) = \mu_{M-2} Q_{M-3}(x) - (\lambda_{M-2}+\mu_{M-2}) Q_{M-2}(x) + \lambda_{M-2} Q_{M-1}(x)
\end{aligned}
$$

We see that only the last component of this relation does not define a new polynomial. For the birth-death process with an infinite number of states the matrix A is infinite and (2.2.11a) for every x defines an eigenvector $\{Q_o(x), Q_1(x), \ldots\}$. These eigenvectors play a role in constructing $\{P_{ij}(t)\}$ which is similar to the role of the eigenvectors of the infinitesimal matrix in constructing $\{P_{ij}(t)\}$ for Markov processes with a finite number of states (See Karlin and Taylor (1975, p. 152)).

We also see that (-x) will be an eigenvalue of A with corresponding eigenvector $\{Q_o(x), Q_1(x), \ldots, Q_{M-1}(x)\}$ if and only if the equation for the last component in (2.2.11) is also satisfied, i.e. if and only if x is a root of equation

$$
(2.2.12) \qquad -xQ_{M-1}(x) = \mu_{M-1} Q_{M-2}(x) - (\lambda_{M-1}+\mu_{M-1}) Q_{M-1}(x)
$$

It was proved by Karlin and McGregor (1957a) that this equation has exactly M roots $x_o, x_1, \ldots, x_{M-1}$ and all of them are positive. By their result

(1957a, p. 494), there exists a positive regular measure $\psi(x)$ on $0 \leq x < \infty$, of total mass one, such that the polynomials $\{Q_o(x),\ldots,Q_{M-1}(x)\}$ are orthogonal with respect to this measure:

$$(2.2.13) \qquad \int_o^\infty Q_i(x)Q_j(x)d\psi(x) = \begin{cases} 0 & \text{for } i \neq j \\ \frac{1}{\pi_i} & \text{for } i = j \end{cases}$$

For a process with a finite number of states the measure ψ will be supported by the roots $x_o, x_1, \ldots, x_{M-1}$ of (2.2.12) only. The probabilities $P_{ij}(t)$ have the following representation:

$$(2.2.14) \qquad P_{ij}(t) = \pi_j \int_o^\infty Q_i(x)Q_j(x)\cdot e^{-tx}d\psi(x), \qquad 0 \leq i,j \leq M-1.$$

This last expression shows that the asymptotic properties of $P_{ij}(t)$ as $t \to \infty$ depend exclusively on the smallest root x_o of (2.2.12). It is also known, that x_o is not a root of any of the polynomials $Q_i(x)$ for $i \leq M-1$ and that x_o is always a simple root of (2.2.12) (See Keilson (1965, p. 417)).

Further, the argumentation of Karlin and McGregor (1957a, Theorem 11) can be used to show that

$$(2.2.15) \qquad \lim_{t\to\infty} \frac{P_{ij}(t)}{P_{i\ell}(t)} = \frac{\pi_j Q_j(x_o)}{\pi_\ell Q_\ell(x_o)}$$

the convergence being exponentially fast.

Now, using (2.2.9) we can prove the following statement.

<u>Statement 2.2.1</u> The conditional probability $r_{i,j}(t)$ tends (as $t \to \infty$) to a positive limit r_j which is independent of i:

$$\lim_{t\to\infty} r_{i,j}(t) = \frac{\pi_j Q_j(x_o)}{\sum_{\ell=o}^{M-1} \pi_\ell Q_\ell(x_o)} = r_j \quad (j = 0,1,\ldots,M-1), \tag{2.2.16}$$

and the rate of convergence is exponential.

□

Using the definition of the polynomials $Q_i(x)$ we easily show that $\{r_o, r_1, \ldots, r_{M-1}\}$ is a left eigenvector of A, corresponding to the eigenvalue $(-x_o)$. In fact, the left eigenvector, corresponding to eigenvalue $(-x_i)$ $(i = 0,1,\ldots,M-1)$ is $\{r_o^{(i)}, r_1^{(i)}, \ldots, r_{M-1}^{(i)}\}$, where

$$r_j^{(i)} = \frac{\pi_j Q_j(x_i)}{\sum_{\ell=o}^{M-1} \pi_\ell Q_\ell(x_i)} \qquad (j = 0,1,\ldots,M-1)$$

(see Appendix F) but only for i=0 is a probability vector obtained. The stated facts may be used to show (See Darroch and Seneta (1965, p.91)) that this vector has the following characteristic property of a stationary distribution.

Statement 2.2.2 If the initial probability distribution is $\{r_o, r_1, \ldots, r_{M-1}\}$, and $r_j(t)$ is the probability of finding the cell in state j at time t given that the cell is not in state M at time t then

$$r_j(t) \equiv r_j \quad , \quad t \geq 0 \tag{2.2.17}$$

□

Finally, this last statement shows together with (2.2.8) that if $\{r_o, r_1, \ldots, r_{M-1}\}$ is the initial distribution, the time to absorption is exponentially distributed with parameter x_o (which is equal to $\lambda_{M-1}\, r_{M-1}$).

2.2.3 Computational aspects

If M is large the computation of the greatest eigenvalue x_o of A using standard matrix operations may be a tedious and time consuming task. Fortunately, in the case of birth-death processes the problem is simplified by the following considerations.

First of all, let us prove some statements about the properties of the polynomials $\{Q_i(x)\}$ defined by (2.2.11).

Statement 2.2.3 $Q_i(0) = 1$ for $i = 0,1,\ldots,M-1$.

Proof The proof follows immediately from the definition of the polynomials (See 2.2.11).

□

Statement 2.2.4 Let us define an additional polynomial $Q_M(x)$ by adding an additional relation to the set (2.2.11a). Then

a) Its roots $0 < x_o < x, <\ldots<x_{M-1}$ are the eigenvalues of A taken with negative sign.

b) There is exactly one root of $Q_{M-1}(x)$ between any two successive roots of $Q_M(x)$.

c) $Q_M(x)$ is convex in the interval $[0,x_o + \varepsilon]$ where ε is some positive number.

Proof a) follows from the definition of the polynomials $Q_i(x)$.

b) can be easily verified by induction.

c) all the roots of $Q_M(x)$ are known to be distinct (See Karlin and McGregor (1957a)). Therefore each interval $[x_j,x_{j+1}]$ $(j = 0,1,\ldots,M-2)$ contains exactly one zero of $Q_M'(x)$. Denote this zero by x_j'. Further, each interval $[x_j',x_{j+1}']$ $(j = 0,1,\ldots,M-3)$ contains exactly one inflection point of $Q_M(x)$. The last considerations show that $Q_M(x)$ is monotonically decreasing and has no inflection points in the interval $[0,x_o+\varepsilon]$ where ε is some positive number. The convexity of $Q_M(x)$ just follows from the fact that $Q_i''(0) > 0$ for $i \geq 2$ (which can be proved by induction).

□

Remark. Of course, a statement of this kind holds for any polynomial $Q_i(x)$ and its roots $(0 < x_o^{(i)} < x_1^{(i)} <\ldots< x_{i-1}^{(i)})$.
The first four polynomials are schematically shown in Fig. 2.2.1.

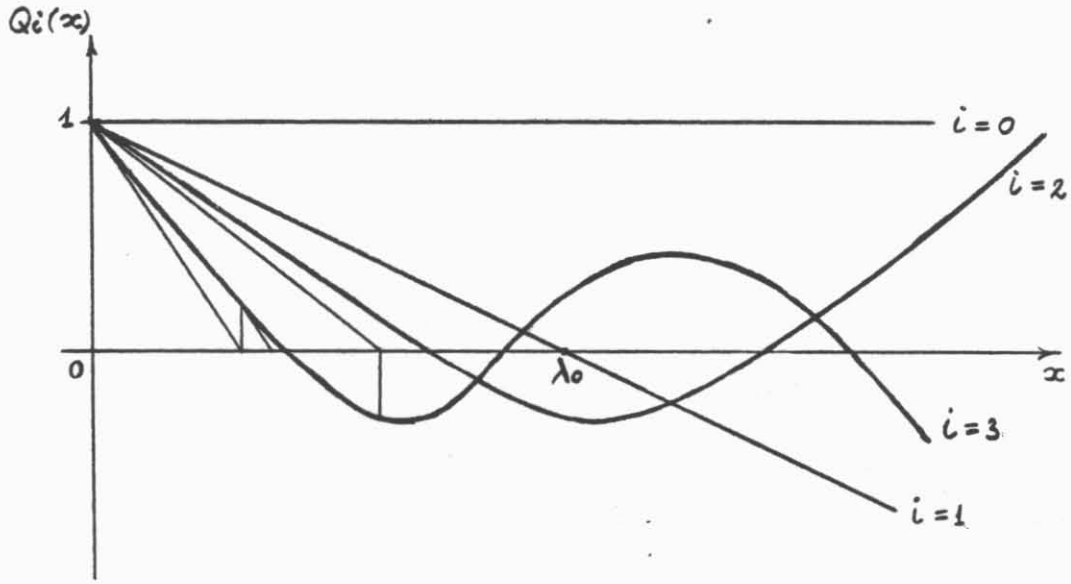


Fig. 2.2.1 The first four polynomials from the system $\{Q_i(x)\}$.

The following Statement shows that the polynomial $Q_i(x)$ is connected with the distribution of the first passage time T_{oi} from the state 0 to the state i.

<u>Statement 2.2.5</u> Denote by $\tilde{f}_{oi}(s)$ the Laplace transform of the first passage time distribution from the state 0 to i.

Then

$$\tilde{f}_{oi}(s) = \frac{1}{Q_i(-s)} \quad ; \qquad i = 1,2,\ldots \tag{2.2.18}$$

<u>Proof</u>: By making i an absorbing state we obtain

$$P\{T_{oi} > t\} = \sum_{j=o}^{i-1} P_{oj}(t) \quad , \tag{2.2.19}$$

where $P_{oj}(t)$ is the transition probability of a process with absorbing state i. Now (2.2.18) can be proved using the spectral representation of $P_{oj}(t)$ and the properties of $\{Q_i(x)\}$. We shall not enter into details here because (2.2.18) can be deduced immediately from the fact that $\tilde{f}_{i-1}(s) = Q_{i-1}(-s)/Q_i(-s)$ which follows from (4.1.3) of Chapter IV ($\tilde{f}_{i-1}(s)$ is the Laplace transform of the first passage time distribution from the state (i-1) to i).

From the fact that

$$(2.2.20) \qquad ET_{oi} = \sum_{m=o}^{i-1} \frac{1}{\lambda_m \pi_m} \sum_{j=o}^{m} \pi_j$$

(See Keilson (1979, p. 61)) and (2.2.18) we obtain

$$(2.2.21) \qquad Q_i'(0) = -ET_{oi}$$

The last equality can be used to obtain a left bound for x_o, the minimal root of $Q_M(x)$ (See Fig. 2.2.1).

$$(2.2.22) \qquad x_o \geq \frac{1}{ET_{o,M}}$$

To calculate x_o we proceed as follows. First of all we verify if $Q_M(1/ET_{o,M-1})$ is negative. If so, we obtain

$$(2.2.23) \qquad \frac{1}{ET_{o,M}} \leq x_o < \frac{1}{ET_{o,M-1}} \quad .$$

The estimate (2.2.22) enables one to obtain an interval of length $(\lambda_{M-1}\pi_{M-1})^{-1}\sum_{j=o}^{M-1}\pi_j$ which certainly includes the value of $(x_o)^{-1}$. A more precise estimate can be obtained using c) of the Statement 2.2.4:

$$(2.2.24)\quad \frac{1}{ET_{o,M}}\left[1+Q_M\left(\frac{1}{ET_{o,M}}\right)\right]<x_o<\frac{1}{ET_{o,M}}+\frac{\left[(\lambda_{M-1}\pi_{M-1})^{-1}\sum_{j=o}^{M-1}\pi_j\right]Q_M\left(\frac{1}{ET_{o,M}}\right)}{Q_M\left(\frac{1}{ET_{o,M}}\right)-Q_M\left(\frac{1}{ET_{o,M-1}}\right)}.$$

Now the precise value of x_o may be obtained quickly using a combination of Newton-Raphson's method and the method of secants (See G. Korn (1968, Chapter 20)).

In the case that $Q_M(1/ET_{o,M-1})$ is positive, we obtain that x_o is greater than $(1/ET_{o,M-1})$. The last value may be used then as a first approximation when calculating x_o by Newton-Raphson's method.

2.3 Relevance of Extreme Value Theory

All the models described before have some common features: the insulator consists of N cells operating independently; the same process of charge accumulation is active in all the cells and the first breakdown occurs after one of the cells reaches a critical state. So, these models work in accordance with the principle of the weakest link: a chain is destroyed as soon as the weakest link in it fails.

Let us consider the regenerative models. In this case after each breakdown the system of cells (insulator as a whole) returns to a fixed

initial state. Let $T_1, T_2, \ldots, T_n$ be the sample of n successive inter-breakdown times. These times are independent identically distributed random variables (assuming $n << N$) and $T = \min(\xi_1, \xi_2, \ldots, \xi_N)$ where ξ_i is the time to explosion of the $i^{\underline{th}}$ cell. To answer the statistical questions formulated in sec. 1.3. we proceed as follows:

a) We express the distribution of the time to explosion in a single cell in the terms of the physical parameters of the model.

b) The distribution law of the interbreakdown time T is expressed using distribution of the time to explosion in a single cell.

c) To estimate the physical parameters of the model we compare the data observed (the set of interbreakdown times $T_1, T_2, \ldots, T_n$) with the distribution law of interbreakdown time predicted by the theoretical model.

Since the number of cells in the models of breakdown is expected to be extremely large, problems arise when trying to express the distribution law of interbreakdown times T in terms of distribution laws of $\xi_1, \xi_2, \ldots, \xi_N$. This problem resembles one of finding the distribution law for $\overline{X} = (X_1 + X_2 + \ldots + X_N)/N$, $X_1, X_2, \ldots, X_N$ being independent and identically distributed random variables. We know that the last problem is solved using the Central Limit Theory, which states, that under rather mild conditions the distribution law of the random variable $\overline{X}$ (after performing an appropriate shift and change of scale) is asymptotically standard normal.

If we are interested in the behaviour of the random variable $T = \min(\xi_1, \ldots, \xi_N)$ rather than that of $\overline{\xi}$, analogous results may be obtained

using Extreme Value Theory instead of Central Limit Theory. The former theory states that under general conditions the random variable T (after appropriate shift and scaling) is distributed in accordance with one of three possible laws as N tends to infinity. Should this law be identified and N be large, the problem of finding the distribution law for T becomes just a problem of finding the appropriate constants of shift and scale (they are called the "normalizing constants" in the literature.)

We see that Extreme Value Theory is relevant for dealing with statistical questions in the case of regenerative models. In the case of non-regenerative models we start our observations at time t=0 and the times of successive breakdowns are $\{\xi_{(1)},\xi_{(2)},\ldots\}$, elements of the ordered sequence of the times to explosion in various cells. (These times are called "order statistics" in the literature). The process of breakdowns in the non-regenerative case is not a renewal process and this complicates the procedure of the related statistical inference. In practical cases we try to answer the statistical questions on the basis of a sample of the first n breakdown times $\{\xi_{(1)},\xi_{(2)},\ldots,\xi_{(n)}\}$. (Note that the distribution of the time of the first breakdown $\xi_{(1)}$ is exactly the same as in the regenerative case). In the analysis of the non-regenerative models we also make use of the Extreme Value Theory which asserts that the random variables $\{\xi_{(1)},\ldots,\xi_{(n)}\}$ (shifted by an appropriate value and inflated by the same scale parameter) behave as they would be realizations of one of three possible well known stochastic processes. Provided we know which of the three relevant processes is appropriate and provided that N is large, statistical analysis of the model may be performed by comparing the observed times of successive breakdowns with the behaviour of breakdown times predicted by the model in consideration.

2.4 Experimental evidence

In this paragraph we show that the "weakest link" models which are described in the present chapter and are object of our investigation have broad experimental support. Consider our system of N cells. Suppose that all the cells are in the same initial state at t=0 and at the same moment a high field F is applied. Similar processes start in every cell and operate independently until the first one explodes. When this happens, the field drops instantly to zero and all the cells rush towards the initial state. However, a common situation is that the time of the field absence is so short that the effect of the breakdown on the non exploded cells may be neglected (we refer to this case as a non-regenerative one). We have pointed out in section 2.1 that the experiment providing data for testing the adequacy of such models should be conducted in a special manner. Since such experiments were not yet conducted, the question about the appropriateness of the non-regenerative models cannot be definitely settled. The exception is the quasi-stationary model which does not require special types of experiments. This model (as we shall see in section 4.1.2) predicts the process of successive breakdowns to be a homogenious Poisson process and there are several sets of data at our disposal which support this prediction (See also Solomon, Klein and Albert (1976)).

Should we artificially preserve the absence of field for a constant time after a breakdown occurs (and then restore its nominal value), the non exploded cells would have returned to the initial state and the regenerative case would be relevant. However, such experiments have not been conducted, so there is no data which enable us to test the adequacy

of the regenerative models (such tests could be based on the comparison of two sample processes of breakdowns: one corresponding to immediate restoration of the field after a breakdown and another - to the delayed restoration). Another way to test the adequacy of the regenerative model is comparing of the sample process of breakdowns in a single specimen, clear of "weak spots" (field is restored immediately after a breakdown occurs) and the times to the first breakdown corresponding to a group of similar cleared specimen (experiments of such kind have also not been performed).

The relevance of the regenerative weakest chain models is supported by data obtained in ramped voltage tests (See Section 1.1). In these types of tests after a breakdown obtains and the field drops to zero, it is restored neither immediately nor after a constant delay (as in previously defined step voltage tests). Instead of this it rises in time at a constant rate r ($F(t) = rt$) until the following breakdown occurs and so on. (The regenerative case might be thought to be relevant in such situations because the field is in the vicinity of zero for a relatively long period of time after a breakdown occurs and the non-exploded cells have time to return to the initial state). As we shall see in Chapter III, the Extreme Value Theory predicts that in the regenerative cases the inter-breakdown times are independent random variables, distributed (after an appropriate linear transformation) identically either in accordance with Weibull's law or in accordance with Gumbel's law (the appropriate distribution functions are given by formulas (3.1.6-3.1.7) dependent on the "thickness" of the left tail of the distribution function of the time to explosion in a single cell.

The Weibull, for shape parameter $\gamma \geq 4$, and Gumbel extreme value distributions are typified by long left tails (negative skewness). Empirical histograms from several ramped voltage experiments also display this property (together with independence of the successive interbreakdown times) thus giving support to "weakest link" breakdown models.

The "weakest link" breakdown mechanism is also supported by the tests, performed on a number of similar specimen (free of weak spots) when only the time to a first breakdown is registered. The times to breakdown in various specimen behave as they would be realizations of independent identically distributed Gumbel or Weibull variates. We consider several examples.

Example 1. In the ramp tests carried out by Osburn and Ormond (1972) on the Si - SiO_2 - Al samples (100 specimens were separately tested) the voltage was linearly raised until the first shorting breakdown occurred. The data consisted of the pairs (V_I, V_F) where V_I is the voltage at which a first non-shorting breakdown occurred (I means "initial") and V_F is the voltage at which a first shorting breakdown occurred (F means "final"). The times to breakdowns of the same type over different samples do not show any significant dependence. However, the data pairs were highly positively correlated and had similarly shaped histograms. The histogram of the voltages V_F is given in Fig. 2.4.1 and chi-square goodness of fit test shows that both Gumbel's and Weibull's laws represent a good fit. The authors didn't try to explain the shape of the histogram but they referred to it as to a typical one and pointed out the inconsistency of the normal fit. Note that in fact the Fig. 2.4.1 represents the histogram of the times to a shorting breakdown since voltage is proportional to time (in ramp tests).

Example 2. Ultrathin samples of Si - SiO_2 - Si (Thickness of the insulator limited to 30 - 300 Å range) were subjected to constant current in the tests performed by E. Harari (1977). Since in such tests the voltage varies to provide a constant current (instead of being constant or ramped) self healing mechanisms cannot be relied upon to neutralize weak spot breakdowns. Therefore the areas of the insulating oxide were kept very small (so as to minimize the probability of occurrence of a weak spot in a given sample). The times to a first breakdown were registered. The typical histogram, based on 100 tested samples is given by Fig. 2.4.3. As we can see, the author tries to fit the normal curve to the observed data. Though the fit in the shown case is not bad (the P-value of the chi-square test is approximately 0.30) the negative skew was typical in twenty experiments under various conditions and this shows the normal fit to be inconsistent. Chi-square tests show that both Gumbel's and Weibull's distribution laws represent a much better fit to the observed data. For example, the P-value equal to 0.60 obtains when fitting the Gumbel's law.

Example 3. In the ramp tests performed by Solomon, Klein and Albert (1976) on the Al^+ - Al_2O_3 - Au^- sample (Al^+ and Au^- mean that aluminium and gold were used as anode and cathode materials respectively) and on the Hf^+ - HfO_2 - Au^- sample the interbreakdown times revealed an independent behaviour and the corresponding histograms were fitted by Gumbel's distribution law (See Fig. 2.4.2. The histogram of voltages to breakdown is plotted, but it corresponds to the histogram of the interbreakdown times).

The authors explain the appearance of the Gumbel's law by a model in which the following assumption has been made: if the field is held constant and equal to F, the process of breakdowns is a homogeneous Poisson process with rate $(1/\tau(F))$ where

$$\tau(F) = \tau(o)e^{-\beta F} \tag{2.4.1}$$

($\tau(o)$ and β are appropriate constants). The probability that a breakdown occurs in the interval (t, t+dt) is equal to

$$p(t)dt = [\tau(o)e^{-\beta rt}]^{-1}dt \cdot \exp\{ - \int_o^t [\tau(o)e^{-\beta rx}]^{-1}dx\} , \tag{2.4.2}$$

so, the distribution function P(t) of the interbreakdown time is

$$P(t) = 1 - \exp \{- \frac{1}{\tau(o)} \int_o^t e^{\beta rx}dx \} = \tag{2.4.3}$$

$$= 1 - \exp\{ - \frac{1}{\beta r\tau(o)} [e^{\beta rt}-1]\} \quad ; \quad t \geq 0 .$$

The linearly transformed interbreakdown time $Y = \beta rT - \ln(\beta r\tau(o))$ has a distribution function equal to

$$P_Y(y) = 1 - \exp\{\frac{1}{\beta r\tau(o)}\} \times \exp\{ - e^y\} \quad ; \quad y \geq -\ln(\beta r\tau(o)), \tag{2.4.4}$$

and, since from the experimental results it follows that $\beta r\tau(o) \approx 100$, Y is approximately Gumbel.

A somewhat different model was used to explain the appearance of Gumbel's law in ramp tests by de Wit, Wienberg and Crevecoeur (1976).

Completely different breakdown mechanisms were used to explain the situations in the examples 1-3. But, as we can see, Gumbel's or Weibull's laws fit the histograms of times to breakdown in every case. It seems very possible that these examples reflect the following reality: a) the insulator consists of a large number of cells; b) the cells are initially in the same state and the same processes develop in every cell independently when the field is applied; c) breakdown occurs after one of the cells reaches some prescribed critical state; d) after a breakdown occurs all the system either returns to the initial state or dies.

Concerning the last example, we could expect that even in the case that the process of breakdowns in step tests is not a Poisson one because of non - regenerativity of the situation after a breakdown occurs), the successive breakdowns in ramp tests will behave as independent identically distributed (in accordance with Weibull's or Gumbel's law) random variables. Indeed, such was the situation which occurred in tests, performed by the same authors when they investigated the dielectric strength of Si - SiO_2 - Au capacitors (See Solomon, Klein and Albert (1976)). In these tests the breakdown processes obtained in step tests even had changes in rate.

Finally, if the process taking place in a single cell when applying step or ramp tests satisfies certain conditions, the assumptions of the model (in Example 3) may be just consequences of a more general "weakest link" model.

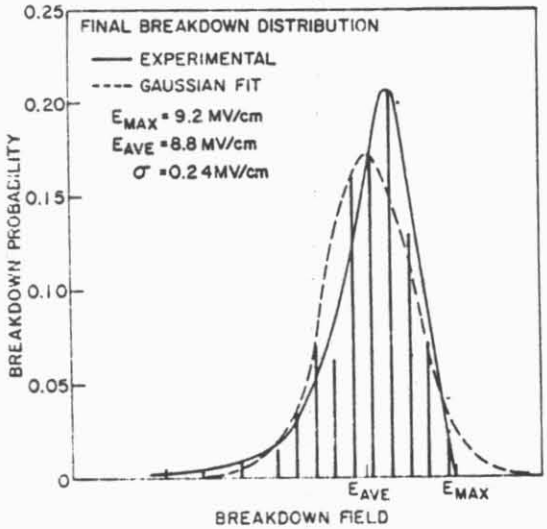


Fig. 2.4.1 Shape of typical final breakdown histogram (from Osburn and Ormond (1972)).

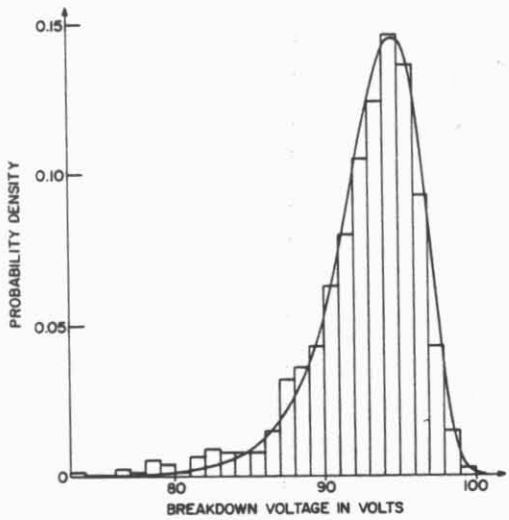


Fig. 2.4.2 Gumbel density function and the histogram of breakdown voltages on the application of rising ramp voltages to an Al-Al_2O_3-Au sample (from Solomon et al (1976)).

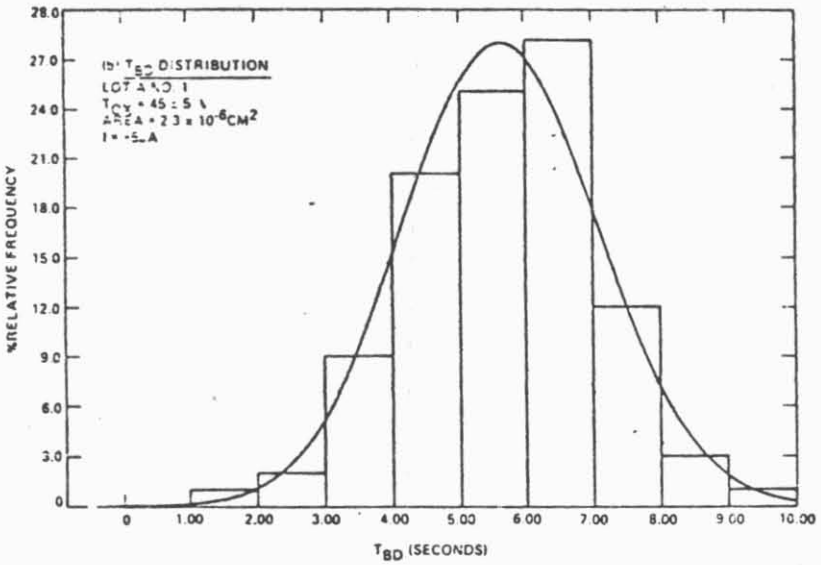


Fig. 2.4.3 Times to a breakdown in ultrathin Si-SiO_2-Si samples subject to constant current (From E. Harari (1978)).

CHAPTER III

3. EXTREME VALUE THEORY - A SURVEY OF RELEVANT RESULTS

As we pointed out in the previous chapter, Extreme Value Theory is crucial for dealing with models of the "weakest link" type (as our models are). In this chapter we formulate the most important (for us) results of this theory in the way that is convenient for applications in our specific models. These results will be referred to often in the following chapters. In the Section 3.1 we formulate the domain of attraction problem for a single extreme variate and review the main results connected with identification of the domain of attraction and finding the normalizing constants (the reader may already have an intuitive explanation of these notions from Section 2.3. In Section 3.2 we reproduce the main results concerning the domain of attraction problem for the joint extreme variates. These results will play a crucial role in analysis of the non-regenerative models. At last, in Section 3.3 we consider the applications of Extreme Value Theory to our models of breakdown phenomena, described in Chapter II. These applications include tests for consistency of the specific model of breakdown with observed data, estimation of the appropriate physical parameters and predicting the types of interdependence between characteristics of the model which is assumed to be operative in a given situation.

3.1 The domain of attraction problem

Let $\xi_1, \xi_2, \ldots, \xi_N$ be independent random variables, identically distributed with common distribution function F. Denote

(3.1.1) $$W_N = \min(\xi_1, \xi_2, \ldots, \xi_N)$$

As well, let x_o be the left endpoint of the distribution function F:

(3.1.2) $$x_o = \inf \{x | F(x) > 0\}$$

(of course x_o is either finite or equal to $(-\infty)$). The distribution function F_N of W_N is given by

(3.1.3) $$F_N(x) = 1 - (1 - F(x))^N \quad , \quad x > x_o$$

and tends to 1 as $N \to \infty$ for every $x > x_o$. However, we can expect that after an appropriate linear transformation the random variable W_N will be asymptotically distributed in accordance with some non-degenerate distribution function G (See Section 2.3).

<u>Definition 3.1.1</u> A distribution function F belongs to the domain of attraction of a non-degenerate distribution function G in the sense of minimum when it is possible to choose two sequences of real numbers $\{b_N\}$ ($b_N > 0$ for $N = 1,2,\ldots$) and $\{a_N\}$ such that for every x

(3.1.4) $$\lim_{N\to\infty} P\{ \frac{W_N - a_N}{b_N} \leq x\} = G(x)$$

The sequences $\{b_N\}$ and $\{a_N\}$ satisfying (3.1.4) are called the sequences of normalizing constants.

□

Identifying the domain of attraction for F and finding the appropriate normalizing constants are the main problems that Extreme Value Theory deals with. So, let us state (without proof) several theorems from this theory. We shall often refer to these theorems throughout the text. Every statement concerning Extreme Value Theory can be formulated in terms of either maximum or minimum. Since we use this theory in the sense of minimum, all the theorems will be formulated in this sense. This means that the asymptotic behaviour of W_N (as $N \to \infty$) will be connected with the left tail properties of F.

First of all, let us define the following three classes of functions

$$(3.1.5) \qquad L_{1,\gamma}(x) = \begin{cases} 1 - e^{-(-x)^{-\gamma}} & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \quad ; \quad \gamma > 0$$

$$(3.1.6) \qquad L_{2,\gamma}(x) \begin{cases} 1 - e^{-x^{\gamma}} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad ; \quad \gamma > 0$$

$$(3.1.7) \qquad L(x) = 1 - e^{-e^{x}} \quad , \quad -\infty < x < \infty.$$

Let us also define a class of regularly varying functions, which play an important role in Extreme Value Theory.

<u>Definition 3.1.2</u> The function F with left endpoint x_o is called a <u>regularly varying</u> function at x_o if it satisfies

$$(3.1.8)\qquad \begin{cases} \lim\limits_{x\downarrow-\infty} \dfrac{F(tx)}{F(x)} = t^{\gamma} & \text{if} \quad x_o = -\infty \\[2ex] \lim\limits_{x\downarrow o} \dfrac{F(x_o+tx)}{F(x_o+x)} = t^{\gamma} & \text{if} \quad x_o \text{ is finite} \end{cases}$$

for every $t > o$ and some finite γ. The number γ is called the <u>exponent</u> of regular variation for F. In the particular case $\gamma = 0$, F is called <u>slowly varying</u> at x_o.

<u>Remark</u>. Condition (3.1.8) is not as strong as it may appear. It is shown (See L. De Haan (1970, p.5)) that if the limit on the left hand side of (3.1.8) exists it must be of the form t^{γ}.

It is possible that F does not belong to the domain of attraction of any non-degenerate function G, but if it does then the normalizing constants may be chosen in such a way that G coincides with one of the three functions $L_{1,\gamma}, L_{2,\gamma}, L$ defined above. Thus, in what follows, G will always denote one of these functions (except where otherwise stated). It is also known that if F belongs to the domain of attraction of G then F belongs to the domain of attraction of any function $G(\beta x+\alpha)$, ($\beta > 0, \alpha$ is real) and of no other function (see Gumbel (1958)). Necessary and sufficient conditions for F to belong to the domain of attraction of $L_{1,\gamma}$, $L_{2,\gamma}$ or L are given by the following theorems. (See Galambos (1978, pp. 49-75) for the proofs.)

<u>Theorem 3.1.1</u> A d.f. F belongs to the domain of attraction of $L_{1,\gamma}$ ($\gamma > 0$) if and only if $x_o = -\infty$ and F varies regularly with exponent $(-\gamma)$ at x_o.

The normalizing constants a_N (shift) and b_N (scale) can be chosen as follows:

$$(3.1.9) \qquad \begin{cases} a_N = 0 \\ b_N = \sup\{t : F(t) \le \frac{1}{N}\} \end{cases}$$

□

<u>Theorem 3.1.2</u> A d.f. F belongs to the domain of attraction of $L_{2,\gamma}(\gamma > 0)$ if and only if its left endpoint x_o is finite and F varies regularly with exponent γ at x_o. The normalizing constants a_N and b_N can be chosen as

$$(3.1.10) \qquad \begin{cases} a_N = x_o \\ b_N = \sup\{t : F(x_o + t) \le \frac{1}{N}\} \end{cases}$$

□

<u>Theorem 3.1.3</u> A d.f. F(t) belongs to the domain of attraction of L if and only if for every t

$$(3.1.11) \qquad \lim_{x \downarrow x_o} \frac{F(x + tr(x))}{F(x)} = e^t ,$$

where

$$(3.1.12) \qquad r(x) = \frac{1}{F(x)} \int_{x_o}^{x} F(y)dy \quad .$$

The normalizing constants a_N and b_N can be chosen as

$$(3.1.13) \qquad \begin{cases} a_N = \sup \{t : F(t) \leq \frac{1}{N}\} \\ b_N = r(a_N) \end{cases}$$

□

Remark (I. Weissman) The expression for b_N in (3.1.13) implies that $b_N = a_N - E(\xi \mid \xi < a_N)$.

The last theorem leads to the following corollary (which plays an important role in applications).

Corollary 3.1.3 Let F be a d.f. with finite left endpoint x_o. If F belongs to the domain of attraction of L with normalizing constants a_N and b_N then

$$\frac{b_N}{a_N - x_o} \to 0 \qquad \text{as} \quad N \to \infty \quad .$$

Proof: The proof follows directly from the fact that

$$\frac{r(t)}{t - x_o} \to 0 \qquad \text{as} \quad t \to x_o$$

(See De Haan (1971,p.82)).

□

Another characterization of the domain of attraction of L is given by the following theorem (see De Haan (1971, p. 80)).

<u>Theorem 3.1.4</u> A d.f. F belongs to the domain of attraction of L with normalizing constants a_N and b_N if and only if

$$\lim_{N\to\infty} N \cdot F(a_N + b_N t) = e^t \tag{3.1.14}$$

for every t.

□

From this theorem the following corollary is easily deduced (see De Haan (1971, p. 58)):

<u>Corollary 3.1.4</u> If the continuous d.f. F belongs to the domain of attraction of L, the normalizing constants can be found from the equations

$$\begin{cases} F(a_N) = \frac{1}{N} \\ F(a_N + b_N) = \frac{e}{N} \ . \end{cases} \tag{3.1.15}$$

□

The following theorem provides other possibilities for the choice of the normalizing constants (see Galambos (1978, p. 61)).

<u>Theorem 3.1.5</u> Let F belong to the domain of attraction of G, the normalizing constants being a_N(shift) and b_N(scale).

Other sets $\{a_N^*\}$ and $\{b_N^*\}$ may also be used as normalizing constants if and only if

$$(3.1.16) \qquad \lim_{N\to\infty} \frac{b_N^*}{b_N} = 1 \; ; \quad \lim_{N\to\infty} \frac{a_N - a_N^*}{b_N} = 0 .$$

□

Let us now define the concept of tail equivalence which is one of the basic concepts in the present work.

<u>Definition 3.1.3</u> Assume that functions F(t) and $F_1(t)$ have a common left hand endpoint x_o. If

$$(3.1.17) \qquad \lim_{t\to x_o} \frac{F(t)}{F_1(t)} = 1,$$

then the functions F(t) and $F_1(t)$ are called <u>tail equivalent</u>.

□

The following theorem shows the importance of this concept to Extreme Value Theory (see Resnick (1971)).

<u>Theorem 3.1.6</u> a) If the distribution functions $F(t)$ and $F_1(t)$ belong to the domain of attraction of G with the same normalizing constants a_N and b_N then,

(3.1.18) (i) $F(t)$ and $F_1(t)$ have a common left endpoint x_o;
(ii) $F(t)$ and $F_1(t)$ are tail equivalent.

b) If the distribution functions $F(t)$ and $F_1(t)$ satisfy (3.1.18) and one of them belongs to the domain of attraction of G with the normalizing constants a_N and b_N then the other one also belongs to the domain of attraction of G with the same normalizing constants a_N and b_N.

□

The next theorem provides a relatively simple way for identifying the domain of attraction of L (see De Haan (1971, p. 110)).

<u>Theorem 3.1.7</u> Suppose that the d.f. $F(t)$ has a positive measurable derivative F' in the neighborhood of x_o.

a) If

$$\frac{F'(x)\cdot\{\int_{x_o}^{x} F(t)dt\}}{[F(x)]^2} \to \quad \text{as} \quad x \to x_o, \tag{3.1.19}$$

then F belongs to the domain of attraction of L.

b) If F' is non-decreasing and F belongs to the domain of attraction of L, then (3.1.19) holds.

□

The next theorem states that multiplication of a distribution function by a regular varying function does not affect its domain of attraction (see L. De Haan (1971, p. 83)).

<u>Theorem 3.1.8</u> Let F_1 and F_2 be distribution functions with common left endpoint x_o. Suppose also that the function

$$U(t) = \frac{F_1(t)}{F_2(t)}$$

is regularly varying at x_o. Then F_1 belongs to the domain of attraction of G if and only if F_2 belongs to the domain of attraction of G.

□

In the models describing the breakdown phenomena F(t) is the distribution function of the time to explosion of a single cell. So x_o, the left endpoint of F will always be finite. Therefore, by the Theorem 3.1.1, F may belong to the domain of attraction of either L or $L_{2,\gamma}$. L. De Haan (1971, p.100) gives a unifying approach which enables one to identify the domain of attraction in this case.

<u>Theorem 3.1.9</u> Suppose F is a distribution function with finite left endpoint x_o. F belongs to the domain of attraction of a non-degenerate d.f. G if and only if for some value of c from (1/2, 1]

$$\lim_{x \downarrow x_o} g(x) = c \tag{3.1.20}$$

where

$$g(x) = \frac{F(x)\{\int_{x_o}^{x}\int_{x_o}^{y} F(t)dtdy\}}{\{\int_{x_o}^{x} F(t)dt\}^2} \quad \text{for all } x > x_o . \tag{3.1.21}$$

If c=1 then G=L. If $c < 1$ then $G = L_{2,\gamma}$ with

$$\gamma = (1-c)^{-1} - 2 \tag{3.1.22}$$

□

Remark a) It is known (See De Haan (1971, p.102)) that if the limit c in (3.1.20) exists, then $c \in (1/2, 1]$.

b) Note that g(x) is similar to the left hand side of (3.1.19). However, the Theorem 3.1.9 does not require the existence of F' in the neighborhood of x_0 as Theorem 3.1.7 does.

Finally, in Fig. 3.1.1 the density functions corresponding to L and $L_{2,\gamma}$ (for several values of γ) are plotted. We shall see that these densities give the main features of the interbreakdown time distribution for the regenerative models.

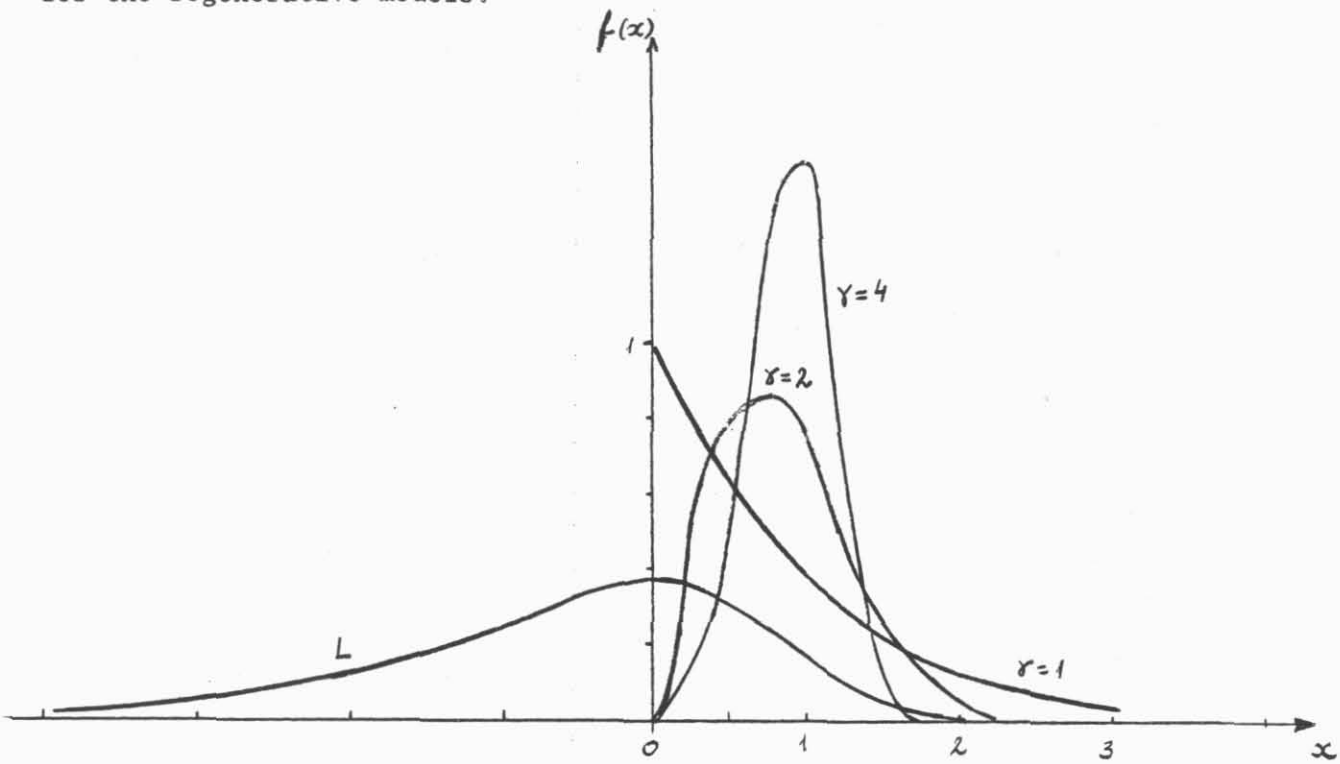


Fig. 3.1.1 The density functions corresponding to L and $L_{2,\gamma}$

As we see from Theorem 3.1.5 we have a large freedom in choosing the normalizing constants so that the convergence in (3.1.4) holds. However, the rate of convergence may depend essentially on the particular choice of the sequences $\{a_N\}$ and $\{b_N\}$. The following theorem (Galambos (1978, p.113)) enables one to estimate the rate of convergence as soon as the normalizing sequences have been chosen.

Theorem 3.1.10 Let G be one of the possible extreme value distributions (in the sense of minimum). Let F belong to the domain of attraction of G. For given sequences $\{a_N\}$ and $\{b_N\}$ $(b_N > 0)$ we put

$$z_N(x) = NF(a_N+b_Nx) \tag{3.1.23}$$

and, for x's for which $G(x) < 1$,

$$\rho_N(x) = z_N(x) + \ln[1-G(x)]. \tag{3.1.24}$$

If x is such that $G(x) < 1$ and if $z_N(x)/N \leq 1/2$, then

$$\left|P\{\frac{W_N-a_N}{b_N} \leq x\} - G(x)\right| \leq [1-G(x)][r_{1,N}(x)+r_{2,N}(x)+r_{1,N}(x)r_{2,N}(x)], \tag{3.1.25}$$

where

$$r_{1,N}(x) = \frac{2z_N^2(x)}{N} + \frac{2z_N^4(x)}{N^2} \cdot \frac{1}{1-q} \quad , \tag{3.1.26}$$

$$(3.1.27) \qquad r_{2,N}(x) = |\rho_N(x)| + \frac{1}{2}\rho_N^2(x) \cdot \frac{1}{1-s},$$

with $q < 1$ and $s < 1$ such that $(2/3) \cdot z_N^2(x)/N \le q$ and $(1/3)|\rho_N(x)| \le s$, respectively.

□

We close this review with a theorem which gives sufficient conditions for convergence of moments of $(W_N - a_N)/b_N$ to the corresponding moments of the limiting extremal distribution (see Pickands III (1968, p. 882)).

<u>Theorem 3.1.11</u> Suppose that F is a d.f. with finite left endpoint belonging to the domain of attraction of $\tilde{G}$ with normalizing constants $\{a_N\}$ and $\{b_N\}$.

Let m be any real positive number. If for some positive integer $N, E(W_N)^m$ is finite then

$$(3.1.28) \qquad \begin{aligned} \lim_{N\to\infty} E\left|\left(\frac{W_N - a_N}{b_N}\right)^-\right|^m &= \int_{-\infty}^{0} (-x)^m \, d\tilde{G}(x), \\ \lim_{N\to\infty} E\left|\left(\frac{W_N - a_N}{b_N}\right)^+\right|^m &= \int_{0}^{\infty} x^m \, d\tilde{G}(x), \end{aligned}$$

where

$$(y)^+ = \begin{cases} y, & y \ge 0 \\ 0, & y < 0 \end{cases}; \quad (y)^- = \begin{cases} 0, & y \ge 0 \\ -y, & y < 0 \end{cases}$$

□

<u>Remark 1</u>. Since $|y|^m = (y^+)^m + (y^-)^m$, under the conditions of Theorem 3.1.11,

$$(3.1.29) \qquad \lim_{N\to\infty} E\left|\frac{W_N - a_N}{b_N}\right|^m = \int_{-\infty}^{\infty} |x|^m \, d\tilde{G}(x).$$

Remark 2. If the conditions of Theorem 3.1.11 are satisfied with m a positive integer, we obtain that

$$(3.1.30) \qquad \lim_{N\to\infty} E\left(\frac{W_N - a_N}{b_N}\right)^m = \int_{-\infty}^{\infty} x^m \, d\,\tilde{G}(x).$$

(This relation follows from (3.1.28) and the identity $y^m = (y^+)^m + (-1)^m \cdot (y^-)^m$).

3.2 The joint distribution of extreme order statistics

In this section we consider the domain of attraction problem for the k first sample extremes. The stated facts will play an important role in analyzing the non-regenerative models of breakdown phenomena.

Let $\xi_1, \xi_2, \ldots, \xi_N$ be independent random variables, identically distributed with common distribution function F (in the models of breakdown ξ_i is the length of life of the $i^{\underline{th}}$ cell). Let us suppose that the left endpoint of F is 0. This assumption simplifies the expressions and is by no means restrictive as will be seen later. Let $\xi_{1N}, \xi_{2N}, \ldots, \xi_{NN}$ be the ordered sequence of the random variables $\xi_1, \ldots, \xi_N$.

Consider the k first sample extremes $\xi_{1N}, \xi_{2N}, \ldots, \xi_{kN}$. It follows from Extreme Value Theory that the asymptotic joint distribution of these variables (after being linearly transformed) may be only of certain specific forms. These forms, as we shall see, depend only on the domain of attraction of F(t) in the sense of minimum (see Sec. 3.1).

Let us suppose in what follows that F belongs to the domain of attraction of G in the sense of minimum the normalizing constants being a_N (shift) and b_N (scale). From the general theory of extreme values we know, that G must coincide either with L or with $L_{2,\gamma}$ (see Theorem 3.1.1).

Moreover, we know (Theorem 3.1.2) that in the case that F belongs to the domain of attraction of $L_{2,\gamma}$, the shift parameter a_N may be taken equal to 0. Since from

$$(3.2.1) \qquad \lim_{N\to\infty} P\{\frac{\xi_{1N}}{b_N} < t\} = L_{2,\gamma}(t)$$

it follows that

$$(3.2.2) \qquad \lim_{N\to\infty} P\{\frac{\ln\xi_{1N} - \ln b_N}{1/\gamma} < t\} = L(t),$$

only the case

$$(3.2.3) \qquad G(t) = L(t) = 1 - \exp\{-e^t\} \quad (-\infty < t < \infty)$$

needs to be considered. So, we suppose that (3.2.3) holds.
Let us shift all the variables $\xi_{1N}, \xi_{2N}, \ldots, \xi_{kN}$ by a_N and scale them by b_N. The transformed variables are

$$(3.2.4) \qquad m_{1N} = \frac{\xi_{1N} - a_N}{b_N}, \ \ldots, \ m_{kN} = \frac{\xi_{kN} - a_N}{b_N}$$

It can be shown (See Weissman (1975)) that not only m_{1N} tends to a non-degenerate random variable but also for fixed k

$$(3.2.5) \qquad \{m_{1N}, m_{2N}, \ldots, m_{kN}\} \to \{m_1, m_2, \ldots, m_k\}, \quad \text{as} \quad N \to \infty$$

in distribution. The density of m_i is

$$(3.2.6) \qquad \phi_i(t) = \frac{1}{(i-1)!} \exp\{-e^t + it\}, \quad -\infty < t < \infty,$$

and the joint density of $\{m_1, m_2, \ldots, m_k\}$ is

$$(3.2.7) \qquad \phi(t_1, t_2, \ldots, t_k) = \exp\{-e^{t_k} + \sum_{i=1}^{k} t_i\}, \quad t_1 \le t_2 \le \ldots \le t_k.$$

The joint density of m_k and the spacings $D_i = m_{i+1} - m_i$ $(i = 1, 2, \ldots, k-1)$ can be shown to be

$$(3.2.8) \qquad \psi(t_k, d_1, d_2, \ldots, d_{k-1}) = \phi_k(t_k) \cdot \prod_{i=1}^{k-1} ie^{-id_i}.$$

Thus, the following result holds (see Weissman (1978)):

<u>Theorem 3.2.1</u> If F belongs to the domain of attraction of L, then the sequence $\{m_1, m_2, \ldots\}$ has the property that for each k $\{m_k, D_1, D_2, \ldots, D_{k-1}\}$ are independent and each D_i is exponentially distributed with mean $(i)^{-1}$.

□

The converse of this theorem is also true (see Weissman (1978, Theorem 4)):

<u>Theorem 3.2.2</u> Suppose that relations (3.2.4)-(3.2.5) define non-degenerate random variables $\{m_1, \ldots, m_k\}$. If m_k and $m_k - m_\ell$ are independent for some $\ell < k$ then F belongs to the domain of attraction of L.

□

Concerning the mean and the variance of m_k, direct calculations show that

$$(3.2.9) \qquad \mu_k = Em_k = \sum_{j=1}^{k=1} \frac{1}{j} - C \approx \ln k \text{ (as k is large)},$$

where C = 0.5772 is Euler's constant and

$$(3.2.10) \qquad \sigma_k^2 = \text{Var } m_k = \frac{\pi^2}{6} - \sum_{j=1}^{k-1} \frac{1}{j^2} \quad .$$

In the case that F belongs to the domain of attraction of $L_{2,\gamma}$ with scaling constant b_N, using (3.2.2) we obtain that for fixed k

$$(3.2.11) \qquad \{\frac{\ln \xi_{1N} - \ln b_N}{1/\gamma}, \ldots, \frac{\ln \xi_{kN} - \ln b_N}{1/\gamma}\} \to \{m_1, \ldots, m_k\}$$

in distribution, as $N \to \infty$. That means that the first k sample extremes displayed using the logarithmic scale of data represent the same asymptotic behavior as the k first extremes in the case that F belongs to the domain of attraction of L.

3.3 Applications to statistical inference

In this section we suppose that the breakdown phenomena is explained by one of the models described in Chapter II. Assume that the model is completely defined after the parameters N, $\theta_1, \theta_2, \ldots, \theta_r$ have been fixed. The time to the explosion ξ_i of the $i^{\underline{th}}$ cell is distributed in accordance with d.f. F(t) (which depends on $\theta_1, \ldots, \theta_r$). Let F belong to the domain of attraction of G in the sense of minimum, the normalizing sequences being $\{a_N\}$ and $\{b_N\}$. We shall show how the parameters of the model may be estimated from a sample $\{T_1, \ldots, T_k\}$ of the first k interbreakdown times.

We know from the previous section that as $N \to \infty$,

$$\{\frac{T_1-a_N}{b_N},\ldots,\frac{T_k-a_N}{b_N}\} \xrightarrow{\text{in distribution}} \{X_1,\ldots,X_k\},$$

where $\{X_1,\ldots,X_k\}$ is a non-degenerate random vector related to the d.f. G. For example, in the case of regenerative models, $X_1,\ldots,X_k$ are iid random variables, distributed in accordance with G. In the case of non-regenerative models and G=L

$$\{X_1,\ldots,X_k\} = \{m_1,m_2,\ldots,m_k\}$$

(see (3.2.5)). Since in our case N is large, we suppose that

$$(3.3.0) \qquad \{T_1,\ldots,T_k\} \overset{d}{=} \{a_N + b_N X_1,\ldots,a_N + b_N X_k\} .$$

Suppose that there are just two parameters describing the model: N and θ. Under the assumption (3.3.0) the estimation procedure for these parameters is as follows:

a) Obtain analytic expressions $a_N(\theta)$, $b_N(\theta)$ of the normalizing constants as functions of N and θ.

b) Find estimates $a_N(\theta)$ and $b_N(\theta)$ using a sample $\{T_1,\ldots,T_k\}$ of interbreakdown times.

c) Equating these estimates to their analytic expressions obtain the two equations - $\begin{cases} a_N(\theta) = a_N(\theta) \\ b_N(\theta) = b_N(\theta). \end{cases}$

The solution of these equations (if it exists) provides the estimators $\hat{N}$ and $\hat{\theta}$.

In the case that there are more than two parameters describing the model, to estimate all of them we use additional samples, corresponding to a variety of experimental conditions (which have a

known effect on the set of parameters $\{N,\theta_1,\ldots,\theta_r\}$).

We shall now discuss several estimation procedures for both regenerative and non-regenerative models.

1. Regenerative models. Suppose that F satisfies the conditions of Theorem 3.1.11 with m=3, so that the first three moments of the normalized sample minimum may be approximated by those of G.*) (Recall that we know a priori that N is large). Denote

(3.3.1) $E = \int x \, d\, G(x)$, $V^2 = \int x^2 \, d\, G(x) - E^2$.

Let $T_1,T_2,\ldots,T_k$ be a sample of interbreakdown times. Denote by $\overline{T}$ and S_T the mean and standard deviation of this sample.

a) Suppose that there are just two parameters describing the model: N and θ (i.e., the underlying d.f. F depends on single parameter θ as well as the appropriate normalizing constants which will be denoted by $a_N(\theta)$ and $b_N(\theta)$). Then, since the first two moments of $(T-a_N(\theta))/b_N(\theta)$ are approximately equal to those of G, one pair of estimators $\hat{N}$ and $\hat{\theta}$ are obtained by using the method of moments; i.e., by solving the equations:

$$\begin{cases} a_N(\theta) + b_N(\theta)\, E = \overline{T} \\ \qquad\qquad b_N(\theta)\, V = S_T, \end{cases}$$

provided the solution exists.

(Recall that the analytic expressions for $a_N(\theta)$ and $b_N(\theta)$ are known). Another pair of estimators can be obtained by using the method of maximum likelihood: MLE's $a_N(\theta)$ and $b_N(\theta)$ are obtained by using the fact that $(T_i-a_N(\theta))/b_N(\theta)$, $i=1,2,\ldots,k$, are i.i.d. random

---

*) We see in the sequel that for the specific class of models considered in the present work all the moments of F exist. (see for example, Karlin (1968, § 3)).

variables distributed in accordance with G (see Johnson and Kotz (1970, Ch. 20,21)). Then if the equations, $a_N(\theta) = a_N(\theta)$, $b_N(\theta) = b_N(\theta)$ have a solution, it represents the MLE's $\hat{N}$ and $\hat{\theta}$.

b) We now discuss the problem of estimating N in the case that the other parameters (describing the properties of the insulating material and the experimental conditions) are known.

The point estimator $\hat{N}$ may be found as a solution of the equation (based on the method of moments)

$$\frac{\overline{T} - a_N}{b_N} = E, \tag{3.3.2}$$

in the case that a solution exists.*)

For obtaining an approximate confidence interval for N as well as for testing hypothesis concerning N the following theorem could be applied.

<u>Theorem 3.3.1</u> Let W_N be a minimum of N i.i.d. random variables with d.f. F. Let F satisfy the conditions of Theorem 3.1.11 with m = 3. Let $W_N^{(1)},\ldots,W_N^{(k)}$ be a sample of k i.i.d. random variables having the same distribution as W_N. Then for every t

$$\left| P\left\{\frac{\overline{W}_N - (a_N + b_N E)}{(b_N V)/\sqrt{k}} \le t\right\} - \Phi(t)\right| \le A(N,k) + B(N,k,t), \tag{3.3.3}$$

where $\overline{W}_N$ is the sample mean, Φ is the d.f. of the standard normal r.v.,

$$A(N,k) = \frac{1.6\,\rho_N}{\sigma_N^3\sqrt{k}},\quad B(N,k,t) = \left|\Phi\left\{\frac{b_N V}{\sigma_N}\left[t - \frac{\sqrt{k}}{V}\left(\frac{\mu_N - a_N}{b_N} - E\right)\right]\right\} - \Phi(t)\right|, \tag{3.3.4}$$

$\mu_N = EW_N$, $\sigma_N^2 = \operatorname{Var} W_N$, $\rho_N = E|W_N - \mu_N|^3$.

---

*) Another possibility is to use the equation $b_N V = S_T$.

Proof: Using the Berry-Esseen Theorem (see Bhattacharya and Rao (1976, p. 104)) we obtain

$$\left|P\left\{\frac{\overline{W}_N-\mu_N}{\sigma_N/\sqrt{k}} \le x\right\} - \Phi(x)\right| \le \frac{1.6\rho_N}{\sigma_N^3\sqrt{k}},$$

for every k and x. Renormalizing $\overline{W}_N$ by using (a_N+b_NE) and (b_NV) instead of μ_N and σ_N, respectively, leads to

$$\left|P\left\{\frac{\overline{W}_N-(a_N+b_NE)}{(b_NV)/\sqrt{k}} \le t\right\} - \Phi\left\{\frac{b_NV}{N}\left[t - \frac{\sqrt{k}}{V}\left(\frac{\mu_N-a_N}{b_N} - E\right)\right]\right\}\right| \le \frac{1.6\rho_N}{\sigma_N^3\sqrt{k}}$$

for every k and t. After adding and substracting $\Phi(t)$ within the the modulus sign we apply the inequality $|a-b| \ge |a|-|b|$ to obtain (3.3.3).

□

The following theorem is related to the behavior of the bound in (3.3.3) as $N\to\infty$.

Theorem 3.3.2 Let the assumptions of Theorem 3.3.1 be satisfied and let A(N,k) and B(N,k,t) be defined by (3.3.4). Then

a) $B(N,k,t) \underset{N\to\infty}{\to} 0$, uniformly in t.

b) $A(N,k) \underset{N\to\infty}{\to} \frac{1.6}{\sqrt{k}} \cdot E\left|\frac{X-E}{V}\right|^3$,

where X is a r.v. distributed in accordance with G.

Proof: To prove a) note that by Theorem 3.1.11,

(3.3.5) $$\lim_{N\to\infty} \sigma_N/b_n = V, \quad \lim_{N\to\infty} \frac{\mu_N-a_N}{b_N} = E.$$

Then, by applying Polya's Theorem (see Rao (1965, p. 100)) we obtain that $B(N,k,t,) \to 0$ as $N\to\infty$, uniformly in t.

To prove b) we note that existence of the limit for (σ_N/b_N) as $N\to\infty$ implies that $\lim_{N\to\infty} (\mu_N-a_N)/b_N$ exists. Thus, μ_N and σ_N can also be used as shift and scale normalizing constants and

$$\lim_{N\to\infty} P\{\frac{W_N-\mu_N}{\sigma_N} \leq x\} = G\ (E+Vx).$$

(See De Haan (1971, Theorem 2.1.2 and Corollary 2.1.3)). So, by Theorem 3.1.11 and (3.1.29)

$$(3.3.6) \qquad A(N,k) = \frac{1.6}{\sqrt{k}}\ E\left|\frac{W_N-\mu_N}{\sigma_N}\right|^3 \underset{N\to\infty}{\longrightarrow} \frac{1.6}{\sqrt{k}}\ E\left|\frac{X-E}{V}\right|^3$$

and b) is proved.

□

Theorems 3.3.1 and 3.3.2 show that for large k and N the d.f. of the statistic

$$Z = \frac{\overline{T} - (a_N+b_N E)}{(b_N V)/\sqrt{k}}$$

can be approximated by Φ and therefore Z may be hopefully used as a pivotal quantity in order to obtain confidence intervals for N or to test hypotheses concerning N.

We also can see, that after the normalizing constants are expressed analytically in terms of $N,\theta_1,\ldots,\theta_r$, the relations (3.3.5) enable us to express the dependence of the mean and standard deviation of the interbreakdown time on the parameters of the model. For example, since the area of insulator, S is proportional to the number of cells N, (3.3.5) enable us to obtain the form of dependence between the mean and standard deviation of the interbreakdown time and S.

2. Non-regenerative models Let ξ_i, the time to explosion in a single cell, be distributed in accordance to d.f. F(t) with left endpoint t=0. Suppose, F belongs to the domain of attraction of L, the normalizing constants being $a_N(\theta_1,\ldots,\theta_r)$ and $b_N(\theta_1,\ldots,\theta_r)$. Suppose we observe the ordered sequence of the first k successive breakdowns $\xi_{(1)},\xi_{(2)},\ldots,\xi_{(k)}$. Since (See paragraph 3.2) N is large (the expected order of N is 10^{10}), the joint distribution of $\xi_{(1)},\ldots,\xi_{(k)}$ is fully determined by the normalizing constants. The questions related to their estimation are discussed in Weissman (1978).

The confidence intervals for a_N and b_N (which were obtained in Weissman (1978) and appear below may be used to construct confidence intervals for the unknown parameters. Since estimation of the normalizing constants a_N and b_N preceeds any other statistical analysis, let us formulate (without proof) the relevant results (which are due to I. Weissman).

First of all, by (3.2.6), the random variables $\{\xi_{(1)},\ldots,\xi_{(k)}\}$ shifted by a_N and scaled by b_N are distributed as $\{m_1,\ldots,m_k\}$. So, a_N and b_N can be estimated using the times of the first k breakdowns. Since the likelihood function for the parameters a_N and b_N is (by 3.2.7)

$$L(a_N,b_N) = \frac{1}{(b_N)^k} \exp\{ -e^{\frac{\xi_{(k)}-a_N}{b_N}} + \sum_{i=1}^{k} (\frac{\xi_{(i)}-a_N}{b_N}) \} \tag{3.3.7}$$

and

$$\sum_{i=1}^{k-1} \xi_{(i)} = (k-1)\xi_{(k)} - \sum_{i=1}^{k-1} i(\xi_{(i+1)}-\xi_{(i)}), \tag{3.3.8}$$

by the factorization criteria the pair $(\xi_{(k)}, \sum_{i=1}^{k-1} \xi_{(i)})$ represents a set of sufficient statistics for (a_N, b_N) as well as the pair $(\xi_{(k)}, \sum_{i=1}^{k-1} i(\xi_{(i+1)} - \xi_{(i)}))$.

Now, the maximum likelihood estimators $\hat{a}_N$ and $\hat{b}_N$ can be obtained using standard arguments:

$$(3.3.9) \qquad \hat{b}_N = \frac{1}{k} \sum_{i=1}^{k-1} (\xi_{(k)} - \xi_{(i)})$$

$$(3.3.10) \qquad \hat{a}_N = \xi_{(k)} - \hat{b}_N \ln k$$

Since

$$(3.3.11) \qquad E\, \hat{b}_N = \frac{b_N}{k} \sum_{i=1}^{k-1} (\mu_k - \mu_i) = \frac{b_N(k-1)}{k}$$

$$(3.3.12) \qquad E\, \xi_{(k)} = a_N + b_N\left(\sum_{i=1}^{k-1} \frac{1}{j} - C\right),$$

and since $\hat{a}_N$ and $\hat{b}_N$ are functions of the pair of sufficient statistics and the completeness condition for this pair is satisfied (See Weissman (1978, Sec. 3)) the minimum variance unbiassed estimators for a_N and b_N are

$$(3.3.13) \qquad \tilde{b}_N = \frac{1}{k-1} \sum_{i=1}^{k-1} (\xi_{(k)} - \xi_{(i)}) = \frac{1}{k-1} \sum_{i=1}^{k-1} i(\xi_{(i+1)} - \xi_{(i)})$$

$$(3.3.14) \qquad \tilde{a}_N = \xi_{(k)} - \tilde{b}_N\left(\sum_{i=1}^{k-1} \frac{1}{j} - C\right)$$

For $i = 1,2,\ldots,k-1$ the random variables $i(\xi_{(i+1)}-\xi_{(i)})$ are independent and identically distributed (exponentially with mean b_N). Further, by Theorem (3.2.1), $\tilde{b}_N$ and $\xi_{(k)}$ are independent. Thus, the variances of $\tilde{b}_N$ and $\tilde{a}_N$ are

$$(3.3.15) \qquad \operatorname{Var} \tilde{b}_N = \frac{b_N^2}{k-1}$$

$$(3.3.16) \qquad \operatorname{Var} \tilde{a}_N = b_N^2 \left\{\frac{1}{k-1}\left(\sum_{i=1}^{k-1} \frac{1}{j} - C\right)^2 + \left(\frac{\pi^2}{6} - \sum_{i=1}^{k-1} \frac{1}{j^2}\right)\right\}$$

The confidence intervals for b_N and a_N may be also obtained using the fact that

$$(3.3.17) \qquad \frac{2(k-1)\tilde{b}_N}{b_N} \sim \chi^2(2(k-1))$$

and that the random variable

$$(3.3.18) \qquad U_k = \frac{\xi_{(k)}-a_N}{b_N} \cdot \frac{b_N}{\tilde{b}_N} = \frac{\xi_{(k)}-a_N}{\tilde{b}_N}$$

is distribution free (See Weissman (1978). In this reference we also find the percentage points of U_k).

Suppose now, that F(t) belongs to the domain of attraction of $L_{2,\gamma}(t)$. In this case we consider the transformed sequence $\{\ln\xi_{(1)},\ln\xi_{(2)},\ldots,\ln\xi_{(k)}\}$ and use (3.2.11). The minimum variance unbiased estimators for $1/\gamma$ and $\ln b_N$ are

$$(3.3.19)\qquad \widetilde{\left(\frac{1}{\gamma}\right)} = \frac{1}{k-1}\sum_{i=1}^{k-1}(\ln\xi_{(k)}-\ln\xi_{(i)}) = \frac{1}{k-1}\sum_{i=1}^{k-1} i(\ln\xi_{(i+1)}-\ln\xi_{(i)}).$$

$$(3.3.20)\qquad \widetilde{(\ln b_N)} = \ln\xi_{(k)} - \widetilde{\left(\frac{1}{\gamma}\right)}\left(\sum_{i=1}^{k-1}\frac{1}{j} - C\right)$$

and the confidence intervals for γ and b_N are obtained via (3.3.17)-(3.3.18). The histogram of logarithmic interbreakdown times $(\ln\xi_{(i+1)}-\ln\xi_{(i)})$ multiplied by i correspond to the exponential law with mean $1/\gamma$.

In the case that F(t) belongs to the domain of attraction of $L_{2,\gamma}(t)$ and its left endpoint is $D > 0$ the histogram of logarithmic shifted interbreakdown times $(\ln(\xi_{(i+1)}-D)-\ln(\xi_{(i)}-D))$ multiplied by i corresponds to the exponential law with mean and standard deviation equal to $1/\gamma$. The estimators for $(1/\gamma)$ and $(\ln b_N)$ when D is known are

$$(3.3.21)\qquad \widetilde{\left(\frac{1}{\gamma}\right)} = \frac{1}{k-1}\sum_{i=1}^{k-1} i\left[\ln(\xi_{(i+1)}-D) - \ln(\xi_{(i)}-D)\right]$$

$$(3.3.22)\qquad \widetilde{(\ln b_N)} = \ln(\xi_{(k)}-D) - \widetilde{\left(\frac{1}{\gamma}\right)}\left(\sum_{i=1}^{k-1}\frac{1}{j} - C\right)$$

In the case that D is not known the situation is much more complicated. Since

$$(3.3.23)\quad \left\{\frac{\ln(\xi_{(1)}-D)-\ln b_N}{1/\gamma},\ldots,\frac{\ln(\xi_{(k)}-D)-\ln b_N}{1/\gamma}\right\}\to\{m_1,\ldots,m_k\}$$

in distribution as $N\to\infty$, the joint distribution of $\{\xi_{(1)},\ldots,\xi_{(k)}\}$ for large N is obtained using (3.2.7):

$$(3.3.24) \qquad L(\xi_{(1)},\ldots,\xi_{(k)};\gamma,\ b_N,\ D) =$$

$$= \begin{cases} \gamma^k \exp\{-\sum_{i=1}^{k} \ln(\xi_{(i)}-D) - \exp\left[\frac{\ln(\xi_{(k)}-D)-\ln b_N}{1/\gamma}\right] + \sum_{i=1}^{k} \frac{\ln(\xi_{(i)}-D)-\ln b_N}{1/\gamma}\}; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad D < \xi_{(1)} \le \ldots \le \xi_{(k)}. \\ 0 \qquad \text{otherwise.} \end{cases}$$

We can see that the maximum likelihood method cannot be applied directly to estimate $\{\gamma, b_N, D\}$ simultaneously since the likelihood function (3.3.24) tends to infinity as $D \to \xi_{(1)}$ and γ is greater than 1. Therefore we shall consider other estimation procedures for D, γ and $(\ln b_N)$.

<u>Estimating D</u> Since $(\xi_{(1)}-D)/b_N$ is distributed in accordance with $L_{2,\gamma}$ and $b_N \to 0$ as $N \to \infty$, we take $\xi_{(1)}$ as an estimator of D: $\hat{D} = \xi_{(1)}$. The mean and standard deviation of D satisfy

$$(3.3.25) \quad \begin{cases} E\hat{D} = D + b_N \ \Gamma(\frac{1}{\gamma} + 1) \to D \quad \text{as} \quad N \to \infty, \\ \sigma_{\hat{D}} = b_N\{\Gamma(\frac{2}{\gamma} + 1) - [\Gamma(\frac{1}{\gamma} + 1)]^2\}^{\frac{1}{2}} \to 0 \quad \text{as} \quad N \to \infty. \end{cases}$$

The problem of finding confidence bounds for D was considered by I. Weissman (see Weissman (1981a, 1981b)). The first paper deals with the case of known shape parameter γ, the second one (unpublished) - with the case of unknown γ.

<u>Estimating γ</u> When D is known, we can estimate γ by means of (3.3.21). When D is unknown we use $\xi_{(1)}$ instead of D in (3.3.21) and obtain the estimator

$$(3.3.27) \quad \widehat{1/\gamma} = \frac{1}{k-2} \sum_{i=2}^{k-1} i\left[\ln(\xi_{(i+1)}-\xi_{(1)}) - \ln(\xi_{(i)}-\xi_{(1)})\right].$$

Let us investigate the mean and the variance of $\widehat{1/\gamma}$.

First of all, note that the distribution of $\widehat{1/\gamma}$ depends on γ only since

$$(3.3.28)\qquad \widehat{1/\gamma} = \frac{1}{k-2}\sum_{i=2}^{k-1} i\left\{\ln\left[\frac{\xi_{(i+1)}-D}{b_N} - \frac{\xi_{(1)}-D}{b_N}\right] - \ln\left[\frac{\xi_{(i)}-D}{b_N} - \frac{\xi_{(1)}-D}{b_N}\right]\right\}$$

and (by 3.3.23) the joint distribution g of $Y_1 = \frac{\xi_{(1)}-D}{b_N},\ldots,Y_k=\frac{\xi_{(k)}-D}{b_N}$ depends on γ only:

$$g(y_1,y_2,\ldots,y_k) = \begin{cases}\gamma^k \exp\{(\gamma-1)\sum_{i=1}^{k} \ln y_i - y_k^{\gamma}\}, & 0 < y_1 \le \ldots \le y_k \\ 0 & \text{otherwise.}\end{cases}$$

The estimator (3.3.27) is, of course, biased since

$$(3.3.29)\qquad \widehat{1/\gamma} \overset{\text{distr.}}{=} \frac{1}{k-2}\sum_{i=2}^{k-1} i\left[\ln Y_{i+1} - \ln Y_i\right] + \frac{1}{k-2}\sum_{i=2}^{k-1} i\left[\ln\left(1-\frac{Y_1}{Y_{i+1}}\right) - \ln\left(1-\frac{Y_1}{Y_i}\right)\right],$$

the mean of the first term is $1/\gamma$ and the mean of the second one (representing the bias) is positive. To investigate the magnitude of the bias we first note the fact that the random variables $(\ln Y_{i+1} - \ln Y_i)$ are independent and distributed exponentially with mean $(i\gamma)^{-1}$. Thus, the Laplace transform, $\tilde{h}(s)$ of the density function h(t) of $(\ln Y_{i+1} - \ln Y_1)$ is given by

$$(3.3.30)\qquad \tilde{h}(s) = \prod_{m=1}^{i} \frac{m\gamma}{m\gamma+s} = \sum_{m=1}^{i} \frac{A_m^{(i)}}{m\gamma+s}$$

and

$$(3.3.30a)\qquad h(t) = \begin{cases}\sum_{m=1}^{i} A_m^{(i)} e^{-m\gamma t}, & t > 0 \\ 0, & \text{otherwise,}\end{cases}$$

where

$$(3.3.31) \qquad A_m^{(i)} = \frac{\gamma(i!)(-1)^{m-1}}{(m-1)!(i-m)!} = \gamma m(-1)^{m-1}\binom{i}{m}.$$

Next we obtain that

$$(3.3.32) \quad E \ln\left(1-\frac{Y_1}{Y_{i+1}}\right) = \sum_{m=1}^{i} A_m^{(i)} \cdot \int_0^\infty \ln(1-e^{-t})e^{-m\gamma t}\, dt =$$

$$= \sum_{m=1}^{i} A_m^{(i)} \int_0^1 (\ln z)(1-z)^{m\gamma-1} dz$$

(Here we used the substitution $1-e^{-t}=z$). The integral in the summand is known (see Gradsteyn and Ryzhik (1965, p. 538)):

$$(3.3.33) \qquad \int_0^1 (\ln z)(1-z)^{m\gamma-1} dz = -\frac{1}{m\gamma}\left[C + \psi(1+m\gamma)\right],$$

where ψ is the so-called psi-function (the log-derivative of the gamma function) and C is Euler's constant ($C \approx 0.577$). Since $\tilde{h}(\alpha) = 1$, we have

$$(3.3.34) \qquad \sum_{m=1}^{i} \frac{A_m^{(i)}}{m\gamma} = 1.$$

Furthermore,

$$(3.3.35) \quad E \ln\left(1-\frac{Y_1}{Y_{i+1}}\right) = -C - \sum_{m=1}^{i} \frac{A_m^{(i)}}{m\gamma}\psi(1+m\gamma), \quad (i=1,2,\ldots).$$

Since $A_m^{(i-1)} = A_m^{(i)} - (m/i)A_m^{(i)}$,

$$(3.3.36) \quad E\left\{i\left[\ln\left(1-\frac{Y_1}{Y_{i+1}}\right) - \ln\left(1-\frac{Y_1}{Y_i}\right)\right]\right\} = -\frac{1}{\gamma}\sum_{m=1}^{i} A_m^{(i)}\psi(1+m\gamma).$$

Thus, by (3.3.29)

$$(3.3.37) \quad E(\widehat{1/\gamma}) = \frac{1}{\gamma} - \frac{1}{\gamma(k-2)}\sum_{i=2}^{k-1}\sum_{m=1}^{i} A_m^{(i)}\psi(1+m\gamma) =$$

$$= \frac{1}{\gamma} - \frac{1}{\gamma(k-2)}\sum_{m=1}^{k-1} \gamma m(-1)^{m-1}\psi(1+m\gamma)\sum_{i=\max(m,2)}^{k-1}\binom{i}{m}.$$

Finally, the identity $\sum_{i=m}^{k-1} \binom{i}{m} = \binom{k}{m+1}$ *) implies that

$$(3.3.38) \quad E(\widehat{1/\gamma}) = \frac{1}{\gamma} - \frac{1}{k-2} \sum_{m=1}^{k-1} m(-1)^{m-1} \psi(1+m\gamma) \left[\binom{k}{m+1} - \delta_{m,1} \right] ,$$

where $\delta_{m,1}=1$ if m=1 and 0 otherwise.

The sum, representing the bias in (3.3.38) possesses some interesting properties from an analytical point of view, which, however, will not be discussed here. We only note that expanding $\psi(1+m\gamma)$ about the point 1 up to two terms, using the fact that $\psi'(1)= -C$ and the two identities

$$(3.3.39) \quad \begin{aligned} &\text{a)} \quad \sum_{m=1}^{k-1} m(-1)^{m-1} \binom{k}{m+1} - 1 \equiv 0, \ k \geq 2 \\ &\text{b)} \quad \sum_{m=1}^{k-1} m^2(-1)^{m-1} \binom{k}{m+1} - 1 \equiv -2, \ k \geq 3^{**)} , \end{aligned}$$

we obtain that

$$(3.3.40) \quad \frac{E(\widehat{1/\gamma}) - 1/\gamma}{1/\gamma} = \frac{C\gamma^2}{k-2} + O(\gamma^3), \ \gamma \to 0.$$

Thus, the relative bias is small for small values of γ.

When γ is large, so is the bias, as the following argument shows. We know (see Gradsteyn and Ryzhik (1965, p.943)) that

$$(3.3.41) \quad \begin{aligned} \psi(1+m\gamma) &= \frac{1}{m\gamma} + \psi(m\gamma) = \\ &= \frac{1}{m\gamma} + \ln(m\gamma) + \int_0^\infty e^{-tm\gamma} \left\{ \frac{1-t-e^{-t}}{t(1-e^{-t})} \right\} dt = \\ &= \frac{1}{m\gamma} + \ln(m\gamma) - \frac{1}{2m\gamma} + O\left(\frac{1}{\gamma^2}\right), \quad \gamma \to \infty \quad . \end{aligned}$$

---

*) This identity can be easily proved by induction.

**) (3.3.39a) can be found in Gradsteyn and Ryzhik (1965, p.3). (3.3.39b) is easily proven by induction as well as the identity (3.3.41a) which appears later in the text.

(see Appendix C). Thus, by (3.3.39a) and identity

$$(3.3.41a) \qquad \sum_{m=1}^{k-1} (-1)^{m-1} \binom{k}{m+1} - 1 \equiv k-2, \; k \geq 2,$$

$$(3.3.42) \qquad E(\widehat{1/\gamma}) - 1/\gamma = C(k) - \frac{1}{2\gamma} + O(\frac{1}{\gamma^2}), \; \gamma \to \infty ,$$

where

$$C(k) = \frac{1}{k-2} \sum_{m=2}^{k-1} m(-1)^m (\ln m) \binom{k}{m+1}.$$

(see Table 3.3.1 for values of C(k)).

Consequently, if it is known a-priori that γ takes on large values, the estimator

$$(3.3.43) \qquad \widetilde{1/\gamma} = 2(\widehat{1/\gamma} - C(k))$$

is less biased than $\widehat{1/\gamma}$ (but has a greater variance). In effect, if it is known a-priori that $\gamma \geq 1$, the likelihood function (3.3.24) is bounded and the MLE of $(D, \gamma, \ln b_N)$ may be obtained by an ordinary procedure of solving the equations obtained by the standard procedure of setting the partial derivatives of L with respect to D, γ and $(\ln b_N)$ to zero. (If these equations have

Table 3.3.1: The values of C(k)

k	C(k)	k	C(k)	k	C(k)	k	C(k)	k	C(k)
1	—	9	0.709	17	0.530	25	0.454	33	0.409
2	—	10	0.674	18	0.518	26	0.447	34	0.404
3	1.386	11	0.644	19	0.506	27	0.440	35	0.400
4	1.125	12	0.618	20	0.496	28	0.434	36	0.396
5	0.976	13	0.596	21	0.486	29	0.429	37	0.392
6	0.878	14	0.577	22	0.477	30	0.423	38	0.388
7	0.807	15	0.560	23	0.469	31	0.418	39	0.385
8	0.753	16	0.544	24	0.461	32	0.413	40	0.381

no solution in the region $0 \leq D \leq \xi_{(1)}$, $\gamma \geq 1$, we should maximize the likelihood function in this region by other means or look for another approach). We will not discuss, however, the properties of estimators produced by this procedure.

To obtain the variance of $\widehat{1/\gamma}$, let us denote the first term in the right hand side of (3.3.29) by A and the second one by B. By Theorem 3.2.1, Var A = $[\gamma^2(k-2)]^{-1}$. Since for i < j

$$(3.3.44)\quad 0 > E\left[(\ln Y_{i+1}-\ln Y_i)\cdot \ln(1-\frac{Y_1}{Y_j})\right] >$$

$$> E\left[(\ln Y_{i+1}-\ln Y_i)\cdot \ln(1-\frac{Y_i}{Y_{i+1}})\right] =$$

$$= \int_0^\infty x\ln(1-e^{-x})\cdot i\cdot\gamma\cdot e^{-i\gamma x}dx \quad > -\infty$$

(see (3.3.33)) and

$$(3.3.45)\quad \ln(1-\frac{Y_1}{Y_{j+1}})-\ln(1-\frac{Y_1}{Y_j}) = \sum_{n=1}^{\infty}\frac{1}{n}\left[(\frac{Y_1}{Y_j})^n-(\frac{Y_1}{Y_{j+1}})^n\right],$$

the dominated convergence principle (see Feller (1971, p. 111)) may be applied to obtain that for i ≤ j

$$(3.3.46)\quad C_{ij}=\mathrm{Cov}\{[\ln Y_{i+1}-\ln Y_i],[\ln(1-\frac{Y_1}{Y_{j+1}})-\ln(1-\frac{Y_1}{Y_j})]\} =$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}\,\mathrm{Cov}\{[\ln Y_{i+1}-\ln Y_i],\ (\frac{Y_i}{Y_{i+1}})^n\left[\prod_{\substack{r=1\\ r\neq i}}^{j-1}(\frac{Y_r}{Y_{r+1}})^n-\prod_{\substack{r=1\\ r\neq i}}^{j}(\frac{Y_r}{Y_{r+1}})^n\right]\}.$$

Furthermore, by (3.3.23) and Theorem 3.2.1,

$$(3.3.47)\quad C_{ij}=\sum_{n=1}^{\infty}\frac{1}{n}\cdot E\left[\prod_{\substack{r=1\\ r\neq i}}^{j-1}(\frac{Y_r}{Y_{r+1}})^n-\prod_{\substack{r=1\\ r\neq i}}^{j}(\frac{Y_r}{Y_{r+1}})^n\right]\cdot \mathrm{Cov}\{(\ln Y_{i+1}-\ln Y_i),(\frac{Y_i}{Y_{i+1}})^n\}=$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}\cdot\left[\prod_{\substack{r=1\\ r\neq i}}^{j-1}\frac{r\gamma}{n+r\gamma}-\prod_{\substack{r=1\\ r\neq i}}^{j}\frac{r\gamma}{n+r\gamma}\right]\cdot\left[\int_0^\infty xe^{-nx-i\gamma x}i\gamma dx-\frac{1}{n+i\gamma}\right] =$$

$$= -\sum_{n=1}^{\infty}\frac{n}{ij\gamma^2(n+i\gamma)}\cdot\prod_{r=1}^{j}\frac{r\gamma}{n+r\gamma}\ .$$

Thus, since $C_{ij}=0$ for $i > j$,

$$(3.3.48)\quad \mathrm{Cov}(A,B)= -\frac{1}{\gamma^2(k-2)^2}\sum_{i=2}^{k-1}\sum_{j=i}^{k-1}\sum_{n=1}^{\infty}\frac{n}{n+i\gamma}\prod_{r=1}^{j}\frac{r\gamma}{n+r\gamma} < 0 .$$

Next we obtain Var B. At first,

$$(3.3.49)\quad e_{i+1}=E\ \ln^2(1-\frac{Y_1}{Y_{i+1}})=\sum_{m=1}^{i}A_m^{(i)}\int_0^1 \ln^2 z\,(1-z)^{m\gamma-1}dz =$$

$$C^2+\frac{\pi^2}{6}+\sum_{m=1}^{i}A_m^{(i)}\cdot\frac{1}{m\gamma}\left[2C\ \psi(1+m\gamma)+\psi^2(1+m\gamma)-\psi'(1+m\gamma)\right]$$

(see (3.3.32), Gradsteyn and Ryzhik (1965, p. 541) and (3.3.34)). Then, by expanding $\ln(1-Y_1/Y_{i+1})$ and using the dominated convergence principle, we obtain:

$$(3.3.50)\quad e_{i,i+1}=E\left[\ln(1-\frac{Y_1}{Y_{i+1}})\ln(1-\frac{Y_1}{Y_i})\right]=-\sum_{n=1}^{\infty}\frac{1}{n}E\left[(\frac{Y_1}{Y_{i+1}})^n\ \ln(1-\frac{Y_1}{Y_i})\right]=$$

$$=-\sum_{n=1}^{\infty}\frac{i\gamma}{n(n+i\gamma)}\sum_{m=1}^{i-1}A_m^{(i-1)}\int_0^{\infty}\ln(1-e^{-t})e^{-nt-m\gamma t}\,dt =$$

$$=\sum_{n=1}^{\infty}\frac{i\gamma}{n(n+i\gamma)}\sum_{m=1}^{i-1}\frac{A_m^{(i-1)}}{n+m\gamma}\left[C+\psi(1+n+m\gamma)\right]$$

and

$$(3.3.51)\quad b_{ii}=E\left[\ln(1-\frac{Y_1}{Y_{i+1}})-\ln(1-\frac{Y_1}{Y_i})\right]^2=e_{i+1}+e_i-2e_{i,i+1}.$$

Next, for $i < j$,

$$(3.3.52)\quad b_{ij}=E\{\left[\ln(1-\frac{Y_1}{Y_{i+1}})-\ln(1-\frac{Y_1}{Y_i})\right]\left[\ln(1-\frac{Y_1}{Y_{j+1}})-\ln(1-\frac{Y_1}{Y_j})\right]\} =$$

$$=\sum_{n=1}^{\infty}\frac{1}{n}E\{\left[\ln(1-\frac{Y_1}{Y_{i+1}})-\ln(1-\frac{Y_1}{Y_i})\right]\left[(\frac{Y_1}{Y_j})^n-(\frac{Y_1}{Y_{j+1}})^n\right]\} =$$

$$=\sum_{n=1}^{\infty}\frac{1}{n}E\{\left[\ln(1-\frac{Y_1}{Y_{i+1}})-\ln(1-\frac{Y_1}{Y_i})\right](\frac{Y_1}{Y_{i+1}})^n\ (\frac{Y_{i+1}}{Y_j})^n\left[1-(\frac{Y_j}{Y_{j+1}})^n\right]\}=$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \cdot \left[\prod_{r=i+1}^{j-1} \frac{r\gamma}{n+r\gamma} \right] \left[1 - \frac{j\gamma}{n+j\gamma} \right] \left[-\sum_{m=1}^{i} \frac{A_m^{(i)}}{n+m\gamma} \left[C + \psi(1+n+m\gamma) \right] + \right.$$

$$\left. + \frac{i\gamma}{n+i\gamma} \sum_{m=1}^{i-1} \frac{A_m^{(i-1)}}{n+m\gamma} \left[C + \psi(1+n+m\gamma) \right] \right].$$

Finally, transforming the last expression with the help of (3.3.30) we obtain

$$(3.3.53) \quad \text{Var } B = \frac{1}{(k-2)^2} \left\{ \sum_{i=2}^{k-1} i^2 b_{ii} + \frac{2}{\gamma} \sum_{i=2}^{k-2} i \sum_{n=1}^{\infty} \left[-\sum_{m=1}^{i} \frac{A_m^{(i)}}{n+m\gamma} \psi(1+n+m\gamma) + \right. \right.$$

$$\left. + \frac{i\gamma}{n+i\gamma} \sum_{m=1}^{i-1} \frac{A_m^{(i-1)}}{n+m\gamma} \psi(1+n+m\gamma) \right] \left[\sum_{j=i+1}^{k-1} \prod_{r=i+1}^{j} \frac{r\gamma}{n+r\gamma} \right] -$$

$$\left. - \left[\sum_{m=1}^{k-1} m(-1)^{m-1} \psi(1+m\gamma) \left(\binom{k}{m+1} - \delta_{m,1} \right) \right]^2 \right\}$$

and as soon as k is given, Var $\widehat{1/\gamma}$ may be calculated as a function of γ by means of the formula

$$(3.3.54) \quad \text{Var } \widehat{1/\gamma} = \frac{1}{\gamma^2 (k-2)} + 2\text{Cov}(A,B) + \text{Var } B$$

with (3.3.48) and (3.3.53) expressing the last two terms. When k is large, the numerical calculations of the above formulae become time-consuming and need to be planned and performed carefully (in particular sums containing $A_m^{(i)}$ must be treated delicately since some terms may reach large values). There are several ways to obtain an upper bound for Var($\widehat{1/\gamma}$) by simpler calculations, (for example, using a finite upper limit instead of ∞ when summing n in (3.3.48) and (3.3.50), applying the inequality $e_{i,i+1} > e_{i+1}$ in (3.3.51), applying Hölder's inequality etc.) However, we have not investigated these, or other, possibilities in detail, and so have no exact information on the behavior of the variance or, indeed, the bias. Thus, to investigate these factors further, we constructed a double pre-

cision FORTRAN program and computed them numerically for particular values of γ and k (see Appendix J, Table J.1). The calculations show that for moderate values of k ($k \le 30$) the estimator $(\widehat{1/\gamma})$ has good properties only when $1/\gamma$ is greater than 0.5. For example, when $1/\gamma=8$ and $k=6$ the bias of $\widehat{1/\gamma}$ is 0.102, the standard deviation is 3.955 and $\sqrt{MSE}$ is 3.954. Should D be known, the estimator (3.3.21) would produce an unbiased estimate with $\sqrt{MSE} = 1/(\gamma\sqrt{k-1}) = 3.577$.

<u>Estimating ($\ln b_N$)</u>. A natural estimator for ($\ln b_N$) can be obtained from (3.3.22) with D replaced by $\xi_{(1)}$:

$$(3.3.56)\quad \widehat{\ln b_N} = \ln(\xi_{(k)}-\xi_{(1)}) - \widehat{1/\gamma}\left(\sum_{i=1}^{k-1}\frac{1}{j}-C\right) =$$

$$\overset{\text{distr}}{=} \ln b_N+\ln Y_k+\ln\left(1-\frac{Y_1}{Y_k}\right) - \widehat{1/\gamma}\left(\sum_{i=1}^{k-1}\frac{1}{j}-C\right)$$

Clearly, the distribution of $(\widehat{\ln b_N}-\ln b_N)$ depends only on γ (a fact which enables the construction of confidence bounds for $\ln b_N$ and D*) in the case that γ is known). By (3.2.9), (3.3.35) and (3.3.37)

$$(3.3.57)\quad E(\widehat{\ln b_N})=\ln b_N - C - \sum_{m=1}^{k-1}\frac{A_m^{(k-1)}}{m\gamma}\psi(1+m\gamma) - \left[E(\widehat{1/\gamma})-1/\gamma\right]\left(\sum_{i=1}^{k-1}\frac{1}{j}-C\right)$$

so that the estimator (3.3.56) is negatively biased. The bias can be investigated for small and large values of γ by the methods we used before, when investigating the bias of $\widehat{1/\gamma}$. Note only, that perhaps it is better to use the estimator $\widetilde{1/\gamma}$ (see (3.3.43)) instead

---

*) Confidence bounds for D may be obtained from the relation: $\ln(\xi_{(k)}-D)-\widehat{\ln b_N} \overset{\text{distr}}{=} -\ln\left(1-\frac{Y_1}{Y_k}\right) + \widehat{1/\gamma}\left(\sum_{i=1}^{k-1}\frac{1}{j}-C\right)$ (see Weissman (1981a) for another approach).

of $\widehat{1/\gamma}$ in (3.3.56) when it is known a-priori that γ is large. (As we have already pointed out, in this case the MLE estimators for the parameters may be found by an ordinary procedure of maximizing the likelihood function. However, even in this case our estimators may be used as first approximation in the maximation procedure.

To calculate $\mathrm{Var}(\widehat{\ln b_N})$ we use the independence of $\ln Y_k$ on the other terms in the right hand side of (3.3.56). To find $\mathrm{Cov}[\ln(1-Y_1/Y_k), \widehat{1/\gamma}]$ we expand $\ln(1-Y_1/Y_k)$ and calculate $\mathrm{Cov}[\ln(1-Y_1/Y_k), A]$ and $\mathrm{Cov}[\ln(1-Y_1/Y_k), B]$ separately, using the same method as in (3.3.46) and (3.3.52). We obtain

$$(3.3.58)\quad v_1 = \mathrm{Cov}(A, -\sum_{n=1}^{\infty} \frac{1}{n}(\frac{Y_1}{Y_k})^n) = \frac{1}{\gamma(k-2)} \sum_{i=2}^{k-1} \sum_{n=1}^{\infty} \frac{1}{n+i\gamma} \prod_{r=1}^{k-1} \frac{r\gamma}{n+r\gamma} \quad ;$$

$$(3.3.59)\quad v_2 = \mathrm{Cov}(B, -\sum_{n=1}^{\infty} \frac{1}{n} (\frac{Y_1}{Y_k})^n) =$$

$$= -\frac{1}{k-2} \sum_{i=2}^{k-1} i \sum_{n=1}^{\infty} \frac{1}{n} E\{\left[\ln(1-\frac{Y_1}{Y_{i+1}}) - \ln(1-\frac{Y_1}{Y_i})\right] \cdot (\frac{Y_1}{Y_{i+1}})^n \cdot (\frac{Y_{i+1}}{Y_k})^n\} -$$

$$- E(B) \cdot E \ln (1- \frac{Y_1}{Y_k}) =$$

$$= -\frac{1}{k-2} \sum_{i=2}^{k-1} i \sum_{n=1}^{\infty} \frac{1}{n} (\prod_{r=i+1}^{k-1} \frac{r\gamma}{n+r\gamma}) \left[-\sum_{m=1}^{i} \frac{A_m^{(i)}}{n+m\gamma} \psi(1+n+m\gamma) + \right.$$

$$\left. + \frac{i\gamma}{n+i\gamma} \sum_{m=1}^{i-1} \frac{A_m^{(i-1)}}{n+m\gamma} \psi(1+n+m\gamma) \right] +$$

$$+ \left[E(\widehat{1/\gamma}) - 1/\gamma \right] \left[C + \sum_{m=1}^{k-1} \frac{A_m^{(k-1)}}{m\gamma} \psi(1+m\gamma) \right] ;$$

where for $i = k - 1$ we define $\prod_{r=i+1}^{k-1}$ as 1. We have also (by 3.2.10)

$$(3.3.60)\quad u_1 = \mathrm{Var}(\ln Y_k) = \frac{1}{\gamma^2} (\frac{\pi^2}{6} - \sum_{j=1}^{k-1} \frac{1}{j^2})$$

and (by (3.3.49))

$$(3.3.61)\quad u_2 = \mathrm{Var}\ \ln\left(1-\frac{Y_1}{Y_k}\right) = e_k - \left[C + \sum_{m=1}^{k-1} \frac{A_m^{(k-1)}}{m\gamma}\ \psi(1+m\gamma)\right]^2 .$$

Thus, the final expression for $\mathrm{Var}\ (\widehat{\ln b_N})$ is

$$(3.3.62)\quad \mathrm{Var}\ (\widehat{\ln b_N}) = u_1+u_2+\left(\sum_{i=1}^{k-1} \frac{1}{j} - C\right)^2 \mathrm{Var}(\widehat{1/\gamma}) - 2\left(\sum_{i=1}^{k-1} \frac{1}{j} - C\right)(v_1+v_2)$$

Finally, we note that $\widehat{1/\gamma}$ and $\widehat{\ln b_N}$ are highly correlated and

$$(3.3.63)\quad \mathrm{Cov}\ (\widehat{1/\gamma}, \widehat{\ln b_N}) = v_1+v_2 - \left(\sum_{i=1}^{k-1} \frac{1}{j} - C\right) \mathrm{Var}(\widehat{1/\gamma}) .$$

This covariance is likely to be negative - we expect that $\ln(1-Y_1/Y_k)$ is close to zero for large values of k so that v_1+v_2 is small compared to $\mathrm{Var}(\widehat{1/\gamma})$.

Calculated values of the bias and standard deviation of $\widehat{\ln b_N}$ for some values of k and γ can be found in Table J.2 of Appendix J. The correlation coefficient between $\widehat{1/\gamma}$ and $\widehat{\ln b_N}$ is also tabulated (see Table J.3).

CHAPTER IV

4. FINITE CHARGE BREAKDOWN MODELS

In this chapter we consider a class of models described in Chapter II namely, finite charge breakdown models. The process of charge accumulation in a given cell is a birth-death process with absorbing state M, "birth" rates $\{\lambda_o, \lambda_1, \ldots, \lambda_{M-1}\}$ and "death" rates $\{\mu_1, \mu_2, \ldots, \mu_{M-1}\}$. As soon as the charge accumulated in one of the cells reaches M, current runaway (breakdown) occurs in this cell after some prescribed time D. In section 4.1 we solve the domain of attraction problem for F(t), the d.f. of the time to explosion of a single cell, and find the normalizing constants. In sections 4.2-4.3 we consider the regenerative and non-regenerative models separately.

4.1 Domain of attraction

In this section we consider two models. In the first model we assume that the initial charge accumulated in every cell at time t=0 is zero. In the second model the initial charge at t=0 is distributed in accordance with the quasi-stationary distribution (See Sec. 2.2).

4.1.1 Zero initial charge

In this model the initial charge of a single cell is zero and F(t) is just a d.f. of the first passage time from 0 to M. To solve the domain of attraction problem we must investigate the left tail properties of F(t).

Let us denote the first passage time from 0 to M by T, the first passage time from the state i to i+1 $(i = 0,1,2,\ldots,M-1)$ by T_i, the corresponding densities by $f(t)$, $f_o(t), f_1(t),\ldots,f_{M-1}(t)$ and their Laplace transforms by $\tilde{f}(s), \tilde{f}_o(s), \tilde{f}_1(s),\ldots,\tilde{f}_{M-1}(s)$.

By the strong Markov property, $T_o, T_1,\ldots,T_{M-1}$ are independent random variables and

$$(4.1.1) \qquad T = T_o + T_1 + \ldots + T_{M-1}$$

As functions $\tilde{f}_o(s),\ldots,\tilde{f}_{M-1}(s)$ satisfy (See Keilson (1965, p.407))

$$(4.1.2) \qquad \tilde{f}_i(s) = \frac{\mu_i + \lambda_i}{\mu_i+\lambda_i+s} \cdot \left\{ \frac{\mu_i}{\mu_i+\lambda_i} \tilde{f}_{i-1}(s)\tilde{f}_i(s) + \frac{\lambda_i}{\mu_i+\lambda_i} \right\} ,$$

they are related by the following recursive formulae :

$$(4.1.3) \qquad \tilde{f}_i(s) = \frac{\lambda_i}{\lambda_i+s+\mu_i(1-\tilde{f}_{i-1}(s))} , \qquad i = 1,2,\ldots,M-1 \quad ;$$

where $\tilde{f}_o(s) = \lambda_o/(\lambda_o+s)$

Since

$$\tilde{f}(s) = \tilde{f}_o(s) \cdot \tilde{f}_1(s) \cdot \ldots \cdot \tilde{f}_{M-1}(s)$$

and

$$(4.1.4) \qquad \tilde{f}_i(s) \sim \frac{\lambda_i}{s} \quad \text{as} \quad s \to \infty \ ,$$

$\tilde{f}(s)$ has at infinity the following asymptotic form

$$(4.1.5) \qquad \tilde{f}(s) \sim \lambda_o \lambda_1 \ldots \lambda_{M-1} \cdot s^{-M} \quad , \qquad s \to \infty \ .$$

(See Keilson (1964) and Rosenlund (1977)).

We see that $\tilde{f}(s)$ varies regularly at infinity, so we can apply one of the Tauberian theorems (See Feller (1971, p. 445)) and conclude that F(t) varies regularly at t=0. The asymptotic expression for F(t) as $t \downarrow 0$ is

$$(4.1.6) \qquad F(t) \sim \lambda_o \lambda_1 \ldots \lambda_{M-1} \cdot \frac{t^M}{M!} \ .$$

We see that the "death" rates μ_i are not represented at all in this asymptotic expression. This means that from the point of view of extreme value theory, the finite birth-death model is equivalent to the finite pure birth model with the same "birth" rates. We can explain intuitively the absence of "death" rates in (4.1.6) as follows: The probability of reaching state M in a short time t with a "death" occurring at some state are negligibly small compared to the probability of reaching state M in time t by "birth" only.

Now let us solve the domain of attraction problem and find the normalizing constants. By Theorem 3.1.2 F(t) belongs to the domain of attraction of $L_{2,M}$ (Weibull distribution function) defined by (3.1.6). The normalizing constant b_N may be found from the equation (due to tail equivalence)

$$(4.1.7) \qquad \lambda_o\lambda_1 \cdots \lambda_{M-1} \frac{t^M}{M!} = \frac{1}{N} .$$

If we suppose the delay D is present, the normalizing constant a_N is equal to D. So, the normalizing constants are

$$(4.1.8) \qquad \begin{cases} a_N = D \\ b_N = [\frac{1}{N} \cdot \frac{M!}{\lambda_o\lambda_1\cdots\lambda_{M-1}}]^{1/M} . \end{cases}$$

4.1.2 Quasi-stationary case

In this model the initial charge at t=0 in a single cell is distributed on the set $\{0,1,\ldots,M-1\}$ in accordance with the quasi-stationary distribution (See Sec. 2.2). The cell explodes as soon as the accumulated charge reaches M and its explosion does not affect the other cells. In the case that the cell explodes with delay D after it reaches state M, the initial distribution is not a quasi-stationary one (since there is a proportion of cells which are in state M). However the process of breakdowns in this case differs from that corresponding to D=0 by a shift in time only. Therefore only the case D=0 needs to be considered.

We have seen from Section 2.2, that if a single cell is "alive" at t=0, the d.f. F_T of the time to explosion T in this cell is

$$(4.1.9) \qquad F_T(t) = 1 - e^{-x_o t} , \quad t \geq 0$$

(where $(-x_o)$ is the maximal eigenvalue of the infinitesimal matrix A). So, F_T belongs to the domain of attraction of $L_{2,1}$ (the standard exponential distribution), the normalizing constants being

$$(4.1.10) \qquad \begin{cases} a_N = 0 \\ b_N = \dfrac{1}{x_o N} \end{cases}$$

Thus, in the quasi-stationary case the process of breakdowns is just a Poisson process with the rate $\lambda = (x_o N)$.

4.2 Regenerative case

As we saw in Section 4.1, in the regenerative case (i.e. in the case that all the cells lose the accumulated charge after a breakdown occurs) F(t) belongs to the domain of attraction of $L_{2,M}$ with normalizing constants given by (4.1.8). So, this model predicts interbreakdown times to come from the three parameter Weibull distribution (these parameters can be estimated using a sample of interbreakdown times). Since the moments of $L_{2,M}$ are given by

$$(4.2.1) \qquad \mu_n = \int_0^\infty x^n \cdot Mx^{M-1}e^{-x^M}dx = \Gamma(\tfrac{n}{M} + 1),$$

denoting the interbreakdown time by T we obtain the following expressions for its mean and standard deviation:

$$(4.2.2) \qquad ET = a_N + b_N\mu_1 = D + [\frac{1}{N} \cdot \frac{M!}{\lambda_o\lambda_1\cdots\lambda_{M-1}}]^{1/M} \cdot \Gamma(\frac{1}{M} + 1)$$

$$(4.2.3) \quad \sigma_T = b_N(\mu_2-\mu_1^2)^{1/2} = [\frac{1}{N} \cdot \frac{M!}{\lambda_o\lambda_1\cdots\lambda_{M-1}}]^{1/M} \cdot [\Gamma(\frac{2}{M} + 1)-\Gamma^2(\frac{1}{M} + 1)]^{1/2}$$

The last formulae show that if the model is valid, we expect the following relations between the sample mean $\overline{T}$ and the sample standard deviation s_T as functions of the area of insulator S :

$$(4.2.4) \qquad \overline{T} = k_1 + k_2S^{-k} \quad ;$$

$$(4.2.5) \qquad s_T = k_3S^{-k} \quad ,$$

where k, k_1,k_2,k_3 are appropriate positive constants with k = 1/M for some positive integer M.

4.3 Non-regenerative case

In this section we consider the methods for estimation of the parameters in the case of non-regenerative models. We consider separately the zero initial charge model and the quasi-stationary model.

4.3.1 Zero initial charge

In this case F(t), the d.f. of the time to explosion of a single cell, belongs to the domain of attraction of $L_{2,M}$ (See Sec. 4.1.1) with normalizing constants given by (4.1.8). Suppose that M is known and $\xi_{(1)},\xi_{(2)},\ldots,\xi_{(k)}$ is the sequence of first k breakdown times. Then the estimator $\hat{D}$ for D is the solution of

$$(4.3.1) \qquad \frac{1}{M} = \frac{1}{k-1}\sum_{i=1}^{k-1} i[\ln(\xi_{(i+1)}-D) - \ln(\xi_{(i)}-D)]$$

and the estimator for $(N \cdot \lambda_o\lambda_1\ldots\lambda_{M-1})$ can be found by solving

$$(4.3.2) \qquad \frac{1}{M}\{\ln(M!) - \ln(N\cdot\lambda_o\lambda_1\ldots\lambda_{M-1})\} = \ln(\xi_{(k)}-\hat{D}) - \frac{1}{M}(\sum_{i=1}^{k-1}\frac{1}{j} - C)$$

(See paragraph 3.3). If the birth rates $\{\lambda_o,\ldots,\lambda_{M-1}\}$ are not known, to estimate N we need an additional sample. This sample must be drawn from another experiment in which the product $\{\lambda_o\lambda_1\ldots\lambda_{M-1}\}$ is changed by a known value.

Supposing that M is unknown, the estimator $\hat{D}$ can be taken equal to $\xi_{(1)}$ (in accordance with the estimating procedure given in the end of Sec. 3). Then the estimators for M and $(N \cdot \lambda_o\lambda_1\ldots\lambda_{M-1})$ are obtained from

$$(4.3.3) \qquad \tilde{M} = \mathrm{int}\{\min\{1, \frac{1}{k-2}\sum_{i=2}^{k-1} i[\ln(\xi_{(i+1)}-\xi_{(1)}) - \ln(\xi_{(i)}-\xi_{(1)})]\}\}^{-1}$$

and (4.3.2) with M replaced by $\tilde{M}$ (int(x) is the integral part of x).

4.3.2 Quasi-stationary case

In this case the process of breakdowns is a homogeneous Poisson process with the rate $\lambda = (x_o N)$. So, if $x_1, x_2, \ldots, x_k$ are the observed interbreakdown times, the product $(x_o N)$ may be estimated by

$$(4.3.4) \qquad \widehat{(x_o N)} = \frac{k}{\sum_{i=1}^{k} X_i}$$

To estimate x_o and N separately we need an additional sample, corresponding to other experimental conditions or some theoretical value of x_o. Note, that just changing of the area of the insulator and observing the process of breakdowns gives us no extra information about N and x_o separately. The computation of x_o via the "birth" and "death" rates may be carried out using the technique of Sec. 2.2.3.

CHAPTER V

5. INFINITE CHARGE BREAKDOWN MODELS - PURE BIRTH CASE

In this chapter we suppose that charge accumulation in a cell proceeds in accordance with a pure birth process. The cell explodes (i.e. current runaway in the cell occurs) as the accumulated charge reaches infinity. There are N cells operating independently and the initial charge in every cell is zero.

Let us denote by ξ_i the time to explosion in the $i^{\underline{th}}$ cell. We see, that $\xi_1,\ldots,\xi_N$ are independent and ξ_i is an infinite sum of independent exponential random variables with parameters $\lambda_o,\lambda_1,\ldots$. In the literature distribution functions of such kind of sums are sometimes called <u>escalator</u> functions (See Chung (1967) and Kingman (1972)). It is known by the theorem of Feller-Lundberg (See Bharucha-Reid (1960, p. 81)) that ξ_i is a finite random variable if and only if

$$\sum_{j=o}^{\infty} \frac{1}{\lambda_j} < \infty \tag{5.1}$$

and this is supposed to hold in what follows.

Let us denote the distribution function of ξ_i by F(t). In order to cope with both regenerative and non-regenerative types of models we must solve the domain of attraction problem for F(t) and to find the appropriate normalizing constants. To do this we first explore the left tail properties of F(t).

In Section 5.1 we consider the domain of attraction problem for F(t) in the sense of minimum. In Section 5.2 we find the appropriate normalizing constants for two important classes of charge accumulation processes. In Section 5.3 the statistical applications of the results are considered.

5.1 Domain of attraction

The Laplace-Stieltjes transform $\tilde{f}(s)$ of F(t) is given by

$$(5.1.1)\quad \tilde{f}(s) = \int_0^\infty e^{-st}\, dF(t) = \prod_{j=0}^{\infty} \left(\frac{\lambda_j}{\lambda_j + s}\right), \quad \text{Re } s \geq 0.$$

Since t=0 is the left endpoint of F and for $s>0$, $x>1$

$$\frac{\tilde{f}(s)}{\tilde{f}(sx)} = \prod_{j=0}^{\infty} \left(1+\frac{(x-1)s}{\lambda_j + s}\right) > (x-1) \sum_{j=0}^{\infty} \frac{s}{\lambda_j + s} \xrightarrow[s\to\infty]{} \infty ,$$

$\tilde{f}(s)$ is not a regularly varying function as $s\to\infty$. Then the Tauberian theorem (see Feller (1971, p. 445)) asserts that F(t) cannot be a regularly varying function as $t \downarrow 0$. So, the Theorem 3.1.1 leads to the conclusion that only two possibilities remain:

a) F belongs to the domain of attraction of L.

b) F does not belong to any domain of attraction.

We shall see that a) obtains by determining the main term of the asymptotic expansion for F(t) as $t \downarrow 0$. This term, by Theorem 3.1.6 may be used to identify the domain of attraction and to find the normalizing sequences $\{a_N\}$ and $\{b_N\}$ as well.

First of all, in Paragraph 5.1.1 we obtain the exact form of F(t) as a limit of a sequence of exponential polynomials. Then we use this form in Paragraph 5.1.2 to obtain the asymptotic representation of F(t) when $\lambda_j = (j+1)^2$, $j = 0,1,\ldots$. This relatively simple case is solved using the theory of Jakobi's θ - function and the asymptotic form of F(t) attained enables one to get a feeling for the possible left tail behaviour of the distribution function F corresponding to power charge accumulation rates ($\lambda_j = a(j+1)^{\alpha}; \alpha > 1$). The last fact makes the power rates case simpler.

Further, in Paragraph 5.1.3 we consider the general pure birth model of charge accumulation. We extend the results of Hirschman and Widder (1955) to obtain the representation of F(t) and its indefinite integrals as $t \downarrow 0$. Then we solve the domain of attraction problem and obtain equations for finding the appropriate normalizing constants. Since these equations are too complicated to give closed expressions for the normalizing constants, in Paragraph 5.1.4 we develop an alternative approach based on the use of Euler's summation method and saddle point methods for finding asymptotic expansions. This approach will be applied in Section 5.2 to find the closed expressions for the normalizing constants $\{a_N\}$ and $\{b_N\}$ in the cases of power and geometrical charge accumulation rates.

5.1.1 Exact form of the distribution function F(t)

First of all, from the form (5.1.1) for the Laplace-Stieltjes transform of F(t) we see that the derivatives $F^{(n)}(t)$ exist for every

integer n and F(t) vanishes at zero together with all its derivatives of finite order, i.e.

$$(5.1.3) \qquad F(0) = 0,\ F'(0) = 0,\ldots,F^{(n)}(0) = 0,\ldots$$

This follows by observing that for every integer k

$$(5.1.4) \qquad s^k\,\tilde{f}(s) = \left(\prod_{j=0}^{k-1} \frac{s\lambda_j}{\lambda_j+s}\right) \cdot \left(\sum_{j=k}^{\infty} \frac{\lambda_j}{\lambda_j+s}\right) \underset{s\to\infty}{\to} 0 .$$

Let us now assume (in the present subsection only) that $\lambda_i \neq \lambda_j$ for $i \neq j$ and denote by $F_n(t)$ the distribution function corresponding to the sum of (n+1) exponential random variables with rates $\lambda_o, \lambda_1,\ldots,\lambda_n$. Let us denote by $f_n(t)$ the density function of $F_n(t)$ and by $\tilde{f}_n(s)$ - the Laplace transform of $f_n(t)$. We know that

$$(5.1.5) \qquad \tilde{f}_n(s) = \prod_{j=o}^{n} \left(\frac{\lambda_j}{\lambda_j+s}\right) = \sum_{j=o}^{n} \frac{A_j(n)}{\lambda_j+s} ,$$

where $A_j(n)$ is given by

$$(5.1.6) \qquad A_j(n) = \lambda_j \left[\prod_{\substack{i=o \\ i\neq j}}^{n} \left(1 - \frac{\lambda_j}{\lambda_i}\right)\right]^{-1}$$

Since (by 5.1.5)) $\sum_{j=o}^{n} A_j(n)/\lambda_j = 1$ for every n, integrating the inversion of $\tilde{f}_n(s)$ from 0 to t we obtain

$$(5.1.7) \qquad F_n(t) = 1 - \sum_{j=o}^{n} \frac{A_j(n)}{\lambda_j} e^{-\lambda_j t}$$

Thus, the exact expression for F(t) is given by

$$(5.1.8) \qquad F(t) = 1 - \lim_{n\to\infty} \sum_{j=o}^{n} \frac{A_j(n)}{\lambda_j} e^{-\lambda_j t},\ t \geq 0.$$

Let us now consider two special cases.

Case a) Let

$$(5.1.9) \qquad \lambda_j = (j+1)^2 \; ; \; j = 0,1,2,\ldots$$

Then (5.1.8) leads to

$$(5.1.10) \qquad F(t) = \begin{cases} 1 + 2 \sum_{j=1}^{\infty} (-1)^j e^{-j^2 t}, & t > 0, \\ 0 & , t \leq 0. \end{cases}$$

(see Karlin (1968, p. 349) for details). The last expression shows that F(t) is a special form of Jakobi's fourth theta-function (see Appendix A).

Case b) In this case

$$(5.1.11) \qquad \lambda_j = b\lambda^j, \; \lambda > 1, \; b > 0, \; j = 0,1,2,\ldots.$$

Formula (5.1.8) takes on the following form:

$$(5.1.12) \qquad F(t) = \begin{cases} 1 - \sum_{j=o}^{\infty} B_j e^{-b\lambda^j t}, & t > 0, \\ 0 & , t \leq 0. \end{cases}$$

where

$$(5.1.13) \qquad B_j = \begin{cases} M \cdot \left[\prod_{i=1}^{j} (1 - \lambda^i) \right]^{-1}, & j > 0, \\ M & , j = 0 \end{cases}$$

with

$$(5.1.14) \qquad M = \left[\prod_{i=1}^{\infty} (1 - \frac{1}{\lambda^i}) \right]^{-1}$$

(see Karlin (1968, p. 350)) for details. It is shown there that

$$(5.1.15) \qquad f(t) = \begin{cases} \sum_{j=o}^{\infty} B_j \cdot b\lambda^j \cdot e^{-b\lambda^j t}, & t > 0, \\ 0 & , t \leq 0. \end{cases}$$

Since $\sum_{j=0}^{\infty} |Bj \cdot b \cdot \lambda^j| < \infty$, we can integrate term by term in (5.1.15). This leads to (5.1.12)).

We see that representation (5.1.8) of F(t) is in general too complicated to be used for solving the domain of attraction problem and we shall look for another approach. Nevertheless, we are able to cope with the first case using some properties of theta-functions. This will be done in the following section.

5.1.2 Quadratic rates

In this paragraph we shall find the asymptotic representation of $F(t)$ in the case $\lambda_j = (j+1)^2$, $j = 0,1,2,\ldots$. In the previous paragraph we found the expression (5.1.10) for $F(t)$ in this case. From this expression we see that

$$F(t) = \theta_4(0|\tau) \ , \tag{5.1.16}$$

where τ is taken to be $\tau = it/\pi$. (see Appendix A). But, using (A.12) we obtain:

$$F(t) = \left(\frac{t}{\pi}\right)^{-1/2} \theta_2\left(0\left|\frac{i\pi}{t}\right.\right) = \tag{5.1.17}$$

$$= 2\sqrt{\frac{\pi}{t}}\, e^{-\pi^2/4t}(1 + e^{-2\pi^2/t} + e^{-6\pi^2/t} + \ldots)$$

Thus, we see that as $t \to 0$,

$$(5.1.18) \qquad F(t) \sim 2\sqrt{\frac{\pi}{t}}\, e^{-\pi^2/4t} \quad .$$

The last asymptotic expression enables us to identify the domain of attraction of F(t) (F(t) belongs to the domain of attraction of L(t)) and to find the normalizing constants. This will be done in Section 5.2. Meanwhile let us note that the factor $1/\sqrt{t}$ in (5.1.18) is a regularly varying one. This factor does not affect the domain of attraction (See Theorem 3.1.8) and, as we shall see later, it does not affect the normalizing scale constant. So it seems quite possible that in the general case of power charge accumulation rates ($\lambda_j = a(j+1)^{\alpha}, \alpha > 1,\ a > 0,\ j = 0,1,2,\ldots$) (F(t) will be of the form

$$(5.1.19) \qquad F(t) \sim \alpha t^{\gamma} e^{-\delta/t^{\beta}} \quad , \quad t \downarrow 0$$

where α, β, γ and δ are constants. We shall see later that this is indeed the form of F(t) in this case.

Let us note that we can obtain the representation (5.1.18) using the Laplace-Stieltjes transform $\tilde{f}(s)$ of F(t)

$$(5.1.20) \qquad \tilde{f}(s) = \int_0^{\infty} e^{-st} dF(t) = \prod_{k=1}^{\infty} \frac{k^2}{k^2+s} =$$

$$= [\prod_{k=1}^{\infty} (1 - \frac{i\sqrt{s}}{k}) e^{i\sqrt{s}/k}]^{-1} \times [\prod_{k=1}^{\infty} (1 + \frac{i\sqrt{s}}{k}) e^{-i\sqrt{s}/k}]^{-1} =$$

$$= s\Gamma(i\sqrt{s})\Gamma(-i\sqrt{s}).$$

The last equality is obtained using the fact that

$$(5.1.21) \qquad \frac{1}{\Gamma(z)} = ze^{Cz} \prod_{k=1}^{\infty} (1 + \frac{z}{k}) e^{-z/k} ,$$

where C is Euler's constant ($C \approx 0.577$).

Since $\Gamma(1 - i\sqrt{s}) = (-i\sqrt{s})\Gamma(-i\sqrt{s})$ and

$$(5.1.22) \qquad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

we obtain the following form for $\tilde{f}(s)$:

$$(5.1.23) \qquad \tilde{f}(s) = \frac{\pi i\sqrt{s}}{\sin(\pi i\sqrt{s})} = \frac{2\pi\sqrt{s}}{e^{\pi\sqrt{s}} - e^{-\pi\sqrt{s}}} =$$

$$= 2\pi\sqrt{s}e^{-\pi\sqrt{s}}[1 + e^{-2\pi\sqrt{s}} + e^{-4\pi\sqrt{s}} + \ldots]$$

Now, if we invert termwise (5.1.23) we obtain the expansion (5.1.17). Such termwise inversion can be justified by applying Theorem 30.1 (see Doetsch (1974, p. 193)). The last expression shows that one way to handle the case of general rates $(\lambda_0, \lambda_1, \ldots)$ would be by trying to represent $\tilde{f}(s)$ by means of a convergent (or asymptotic, for large s) series and then to invert term by term.

5.1.3 General Approach

In this paragraph we shall consider the domain of attraction problem for distribution functions F whose Laplace-Stieltjes transform is given by 5.1.1. The behavior of f(t) as $t \downarrow 0$ was considered by Hirschman and Widder (1955) when investigating f(t) as a kernel of a convolution transform. Their results are presented in the following form (Theorem 3.1, Ch. V):

Denoting for $r > 0$

$$\lambda(r) = \sum_{j=o}^{\infty} (\lambda_j + r)^{-1} \quad ; \tag{5.1.24}$$

$$\sigma(r) = \left(\sum_{j=o}^{\infty} (\lambda_j + r)^2 \right)^{1/2} \quad ; \tag{5.1.25}$$

$$\Lambda(r) = \frac{e^{r\lambda(r)} f(r)}{\sigma(r)} \quad , \tag{5.1.26}$$

the $n^{\underline{th}}$ derivative of $f(t)$ satisfies:

$$f^{(n)}(\lambda(r)) \sim \frac{1}{\sqrt{2\pi}} r^n \Lambda(r) \quad \text{as} \quad r \to \infty \tag{5.1.27}$$

for $n = 0,1,2,\ldots$. $(f^{(o)}(t) \equiv f(t))$. We could use this result to first obtain the asymptotic estimate of $f(t)$ as $t \downarrow 0$ and then, by integrating this asymptotic relation, obtain the representation of $F(t)$ as $t \downarrow 0$. Instead of doing so we show that (5.1.27) remains true for n negative where we interpret $f^{(-n)}(t)$ via

$$f^{(-n)}(t) = \int_o^t f^{(-n+1)}(x)dx \quad ; \quad n = 1,2,\ldots \tag{5.1.28}$$

<u>Lemma 5.1.3</u> Let n be a positive integer. Then

$$f^{(-n)}(\lambda(r)) \sim \frac{1}{\sqrt{2\pi}} r^{-n} \Lambda(r) \text{ as } r \to \infty . \tag{5.1.29}$$

<u>Proof</u>: The proof is a modified version of Hirschman and Widder's proof. First of all, let us note that F(t) vanishes at zero together with all its derivatives. Further, since all the poles of $\tilde{f}(s)$ are in the left half-plane, applying the inversion formula (See Doetch (1974)) we obtain

$$f^{(-n)}(u) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} s^{-n} e^{su} \tilde{f}(s) ds, \quad r > 0. \tag{5.1.30}$$

After a change of variable $s = r-p/\sigma(r)$, we have

$$f^{(-n)}(u) = \frac{1}{2\pi i} \frac{e^{ru} r^{-n}}{\sigma(r)} \int_{-i\infty}^{i\infty} [1 - \frac{p}{r\sigma(r)}]^{-n} \exp[- \frac{pu}{\sigma(r)}] \tilde{f}(r - \frac{p}{\sigma(r)}) dp. \tag{5.1.31}$$

But we may easily verify that

$$\frac{\tilde{f}(r - \frac{p}{\sigma(r)})}{\tilde{f}(r)} = \prod_{j=o}^{\infty} [1 - \frac{p}{\sigma(r)(\lambda_j + r)}]^{-1} = \tag{5.1.32}$$

$$= e^{p\lambda(r)/\sigma(r)} [E_r(p)]^{-1} ,$$

where

$$E_r(p) = \prod_{j=o}^{\infty} \{[1 - \frac{p}{\sigma(r)(\lambda_j + r)}] e^{\frac{p}{\sigma(r)(\lambda_j + r)}} \} . \tag{5.1.33}$$

If we set $u = \lambda(r)$ in (5.1.31), we obtain

$$f^{(-n)}(\lambda(r)) = r^{-n}\Lambda(r)I_r. \tag{5.1.34}$$

where $\Lambda(r)$ is defined by (5.1.26) and

$$I_r = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} [1 - \frac{p}{r\sigma(r)}]^{-n}[E_r(p)]^{-1}dp\,. \tag{5.1.35}$$

Now, let us show that

$$\lim_{r\to\infty} E_r(p) = e^{-p^2/2} \tag{5.1.36}$$

for all p, uniformly for p in any compact set.

First of all let us note that

$$|\ln[(1-p)e^{p+p^2/2}]| \leq 2|p|^3, \quad |p| \leq 1/2, \tag{5.1.37}$$

where the main branch of logarithm is taken.

The proof of this inequality can be found in Titchmarsh (1939, p. 246).

So, since by definition of $\sigma(r)$

$$\frac{p^2}{2} = \frac{1}{2}\sum_{j=o}^{\infty} [\frac{p}{\sigma(r)(\lambda_j+r)}]^2\,, \tag{5.1.38}$$

we obtain

$$|\ln[E_r(p)e^{p^2/2}]| \leq 2\sum_{j=o}^{\infty} \left|\frac{p}{\sigma(r)(\lambda_j+r)}\right|^3 \leq \frac{2|p|^3}{r\sigma(r)} \tag{5.1.39}$$

Then, since

$$(5.1.40) \qquad r\sigma(r) \to \infty \quad \text{as} \quad r \to \infty \quad ,$$

(5.1.36) follows immediately from (5.1.39).

The equality (5.1.36) was used in the above mentioned Theorem 3.1 to prove that I_r tends to $(2\pi)^{-1/2}$ as $r \to \infty$.

Let us prove that the same is true in our case. Take $a > 0$ and write I_r as

$$(5.1.41) \qquad I_r = \frac{1}{2\pi i}\left\{ \int_{-i\infty}^{-ai} + \int_{-ai}^{ai} + \int_{ai}^{i\infty} \right\}$$

Taking $\varepsilon > 0$ we then have

$$(5.1.42) \quad |I_r - (2\pi)^{-1/2}| \leq \frac{1}{2\pi}\Big\{\Big| \int_a^\infty [1 + \frac{it}{r\sigma(r)}]^{-n}(E_r(-it)]^{-1}dt\Big| +$$

$$+\Big|\int_{-a}^{a} \{[1 - \frac{it}{r\sigma(r)}]^{-n}\cdot[E_r(it)]^{-1} - \exp(-t^2/2)\}dt\Big| +$$

$$+ 2\int_a^\infty \exp(-t^2/2)dt + \Big|\int_a^\infty [1 - \frac{it}{r\sigma(r)}]^{-n}[E_r(it)]^{-1}dt\Big|\Big\}$$

Now,

$$(5.1.43) \qquad |E_r(-it)|^2 = |E_r(it)|^2 = \prod_{j=o}^{\infty} \{1 + \frac{t^2}{[\sigma(r)(\lambda_j + r)]^2}\} =$$

$$= 1 + t^2 \sum_{j=o}^{\infty} \frac{1}{[\sigma(r)(\lambda_j+r)]^2} + \frac{1}{2} t^4 \sum_{j=o}^{\infty} [\frac{1}{\sigma(r)(\lambda_j+r)}]^2 \{ \sum_{m=o}^{\infty} \frac{1}{[\sigma(r)(\lambda_m+r)]^2} - \frac{1}{[\sigma(r)(\lambda_j+r)]^2} \} + \dots ,$$

where all the remaining terms are positive. Let us denote

(5.1.44) $$\tilde{\lambda} = \inf(\lambda_1, \lambda_2, \dots)$$

Then, by (5.1.40), it is possible to choose $R > 0$ such that

(5.1.45) $$1 - [\sigma(r)(\tilde{\lambda} + r)]^{-2} > 1/2 \quad \text{for} \quad r > R.$$

Therefore, using the definition of $\sigma(r)$, we have as $r > R$,

(5.1.46) $$|E_r(-it)|^2 > 1 + t^2 + \frac{1}{4}t^4 \quad .$$

So,

(5.1.47) $$\left| \int_a^{\infty} [1 \pm \frac{it}{r\sigma(r)}]^{-n} [E_r(\pm it)]^{-1} dt \right| <$$

$$< \int_a^{\infty} \{1 + t^2[r\sigma(r)]^{-2}\}^{-n/2} (1 + t^2 + \frac{1}{4} t^4)^{-1/2} dt \leq$$

$$\leq \int_a^{\infty} (1 + t^2 + \frac{1}{4}t^4)^{-1/2} \, dt < \pi\varepsilon/2 \quad \text{whenever} \quad a > a_1(\varepsilon)$$

and also

$$(5.1.48) \qquad 2\int_a^\infty \exp(-t^2/2)\,dt < \pi\varepsilon/2 \quad \text{whenever} \quad a > a_2(\varepsilon) .$$

Denoting $a(\varepsilon) = \max[a_1(\varepsilon), a_2(\varepsilon)]$ we obtain from (5.1.36) that

$$(5.1.49) \quad \left| \int_{-a(\varepsilon)}^{a(\varepsilon)} \{[1 - \frac{it}{r\sigma(r)}]^{-n}[E_r(it)]^{-1} - \exp(-t^2/2)\}dt \right| < \pi\varepsilon/2 \text{ for } r>R_1(\varepsilon)$$

and therefore, as $r > \max(R,R_1(\varepsilon))$

$$(5.1.50) \qquad |I_r - (2\pi)^{-1/2}| < \varepsilon$$

This completes the proof.

□

We are now in a position to solve the domain of attraction problem and to provide some indirect information concerning the nature of the normalizing sequences $\{a_N\}$ and $\{b_N\}$.

<u>Theorem 5.1.3</u> F belongs to the domain of attraction of L, the sequence of normalizing constants $\{a_N\}$ and $\{b_N\}$ being determined by

$$(5.1.51) \qquad \begin{cases} b_N = \dfrac{1}{r_N} \\ a_N = \lambda(r_N) \end{cases} ,$$

where r_N is the root of the equation

$$(5.1.52) \qquad \frac{1}{\sqrt{2\pi}} \cdot \frac{\Lambda(r)}{r} = \frac{1}{N} .$$

<u>Proof</u>: Let $\lambda^{-1}(t)$ be the inverse function of $\lambda(r)$. By Lemma 5.1.3

$$(5.1.53) \qquad \frac{f(t) \cdot \int_0^t F(x)dx}{[F(t)]^2} \longrightarrow 1 \ , \quad \text{as } t \downarrow 0,$$

and, by the theorem of R. von Mises (See Theorem 3.1.7) F belongs to the domain of attraction of L. Since

$$(5.1.54) \qquad \frac{\int_0^t F(x)dx}{F(t)} \sim \frac{1}{\lambda^{-1}(t)} \ , \quad \text{as} \quad t \downarrow 0 \ ,$$

by Theorem 3.1.3 and Theorem 3.1.5 the normalizing constants can be found from the equations

$$(5.1.55) \qquad \begin{cases} \dfrac{\Lambda(\lambda^{-1}(a_N))}{\sqrt{2\pi}\ \lambda^{-1}(a_N)} = \dfrac{1}{N} \\[2ex] b_N = \dfrac{1}{\lambda^{-1}(a_N)} \end{cases}$$

After we set $r_N = \lambda^{-1}(a_N)$, the theorem is proved.

□

Concerning the numerical aspects of the problem, we note, that in connection with equation (5.1.52) if we set

$$F(t) = \exp(-\chi(t)),\ t > 0$$

then from (5.1.29)

(5.1.56) $\quad \chi'(\lambda(r)) \sim -r, \qquad r\to\infty$.

But we know that if two positive functions f and g satisfy the conditions

$$\lim_{r\to\infty} \int_0^r f(x)dx = \lim_{r\to\infty} \int_0^r g(x)dx = \infty$$

and

(5.1.56a) $\quad \lim_{r\to\infty} \dfrac{f(r)}{g(r)} = c \qquad (0 \le c \le \infty)$

then

$$\lim_{r\to\infty} \frac{\int_0^r f(x)dx}{\int_0^r g(x)dx} = c$$

(see De Haan (1971, p. 12) for a proof of this statement). It is also known that if (5.1.56a) holds with $0 < c < \infty$ then the integrals $\int_0^\infty f(x)dx$ and $\int_0^\infty g(x)dx$ either both converge or both diverge.

Thus after multiplying both sides of (5.1.56) by $\lambda'(r)$ and integrating them from 0 to r we obtain

$$\chi(\lambda(r)) \sim -\int_0^r x\lambda'(x)dx = -r\lambda(r) - \ln \tilde{f}(r),\ r\to\infty \quad .^{*)}$$

Since the normalizing constant a_N can also be found from the equation

(5.1.57) $\qquad \chi(t) = \ln N,$

r_N can be approximated by the root of the equation

(5.1.58) $\qquad -r\lambda(r) - \ln \tilde{f}(r) = \ln N$

This root may be used as a first approximation when solving (5.1.52) by some iterative procedure. However, it would clearly be desirable to have an explicit way of computing the constants, at least for some specific cases. Such a method is developed in the following paragraph.

---

*) Here we identify $f(r)$ with $\lambda'(r)\chi'(\lambda(r))$ and $g(r)$ with $(-r)\lambda'(r)$.

5.1.4 An alternative approach

The method described in the previous section is general in that it deals with any sequence $\{\lambda_j\}$ of rates satisfying (5.1). However, the determination of the sequences $\{a_N\}$ and $\{b_N\}$ is not explicit and so we seek an alternative approach which might be applied successfully for given forms of the dependence of λ_j on j. This approach may also be useful when dealing with infinite convolutions of non-exponential densities. The method is based on:

a) finding a representation of $\tilde{f}(s)$ for large s using summation methods.

b) finding, by saddle point methods, the main term of the asymptotic expansion of F(t) as $t \downarrow 0$.

c) finding the normalizing constants from this main term.

This approach has more chance of producing simple expressions for the normalizing constants since the main term often has a fairly simple form. Concerning the representation of f(s) for large s the idea is as follows. We know that

$$(5.1.59) \qquad -\ln \tilde{f}(s) = - \sum_{j=o}^{\infty} \ln\left(\frac{\lambda_j}{\lambda_j+s}\right) .$$

Now let us suppose for simplicity that $\{\lambda_j\}$ form a monotonically increasing sequence and let us denote by $\lambda(u)$ a continuous function that satisfies

$$(5.1.60) \qquad \lambda(j) = \lambda_j \quad , \qquad j = 0,1,2,\ldots .$$

Then the function $(-\ln \tilde{f}(s))$ can be represented by

$$(5.1.61) \qquad -\ln \tilde{f}(s) = -\int_0^\infty \ln\left(\frac{\lambda(u)}{\lambda(u)+s}\right)du - \frac{1}{2}\ln\frac{\lambda_o}{\lambda_o+s} + R(s).$$

(See Fig. 5.1.4). The left hand side is the total area

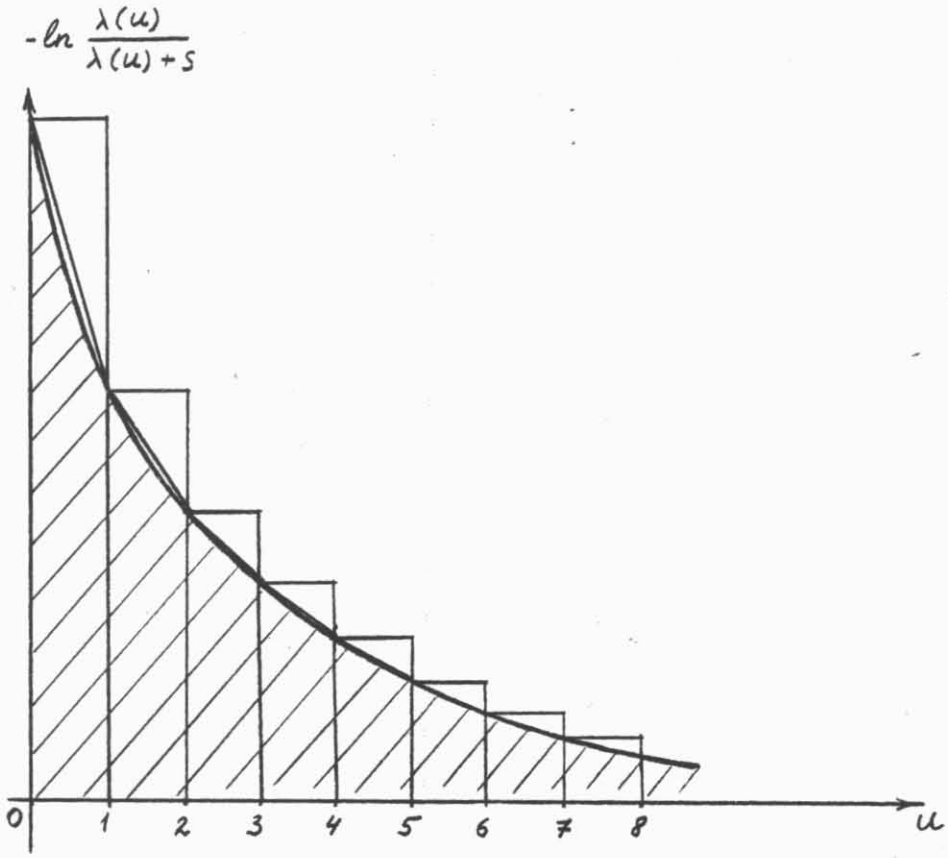


Fig. 5.1.4

under the "stairs", the first term of the right hand side is the shaded area, the second term is the sum of the areas of the non-shaded triangles and R(s) is the remainder. The expansion (5.1.61) may be continued in accordance with Euler's summation formula (See Appendix B), but in many cases already (5.1.61) will be adequate for our purposes (depending on the behaviour of R(s) for large s).

For step b) we use (5.1.61) and obtain the integral representation of the inverse for F(t). Using the saddle point method we obtain the asymptotic representation of F(t) as $t \downarrow 0$ using this integral representation for F(t). The first term of this representation contains all the information needed to find the normalizing sequence (See Theorem 3.1.6).

In the following section we apply this alternative approach for the cases of power and geometric charge accumulation rates. The latter is of particular interest for the breakdown model since electric field calculations indicate that the current through a cell increases geometrically as a function of the number of charge units in that cell.

5.2 Normalizing constants

In this section we apply the alternative approach developed in the paragraph 5.1.4 to solve the domain of attraction problem for two specific cases and to find the normalizing constants. In the first case the pure birth process of charge accumulation in a single cell has birth rates of the type $\lambda_j = a(j+1)^{\alpha}$ (power charge accumulation rates) and in the second one - of the type $\lambda_j = b\lambda^j$ (geometric charge accumulations rates). Let us consider these cases separately.

5.2.1 Power charge accumulation rates

In this paragraph we shall consider the case

$$\lambda_j = a(j+1)^{\alpha} \quad ; \quad j = 0,1,\ldots,\alpha > 1 \quad ,$$

using the alternative approach. We shall compare the results with those obtained by using the general approach.

1. Asymptotic Behavior of the Distribution Function

By applying Euler's summation method (see Appendix B) we obtain

$$(5.2.1) \qquad -\frac{d}{ds}\left[\ln \tilde{f}(s)\right] = \sum_{j=o}^{\infty} \frac{1}{aj^{\alpha}+s} - \frac{1}{s} =$$

$$= \int_{o}^{\infty} \frac{du}{au^{\alpha}+s} + \frac{1}{2s} + R(s) - \frac{1}{s} =$$

$$= \frac{\pi a^{-1/\alpha}}{\alpha \sin(\pi/\alpha)} s^{\frac{1}{\alpha}-1} - \frac{1}{2s} + R(s)$$

The full asymptotic expansion of $\sum_{j=o}^{\infty} (\frac{1}{aj^{\alpha}+s})$ can be found in Hirschman and Widder (1955, p. 118). There it is shown that $R(s) + a\,\zeta(-\alpha)/s^2$ is $O(1/s^3)$ as $s\to\infty$. (ζ is the Riemann zeta-function (see Titchmarsh (1964, p. 152)). So, by integrating (5.2.1) from 1 to s we obtain

$$(5.2.2) \qquad \tilde{f}(s) = \tilde{k}\, \exp\{-\frac{\pi a^{-1/\alpha}}{\sin(\pi/\alpha)} s^{1/\alpha} + \frac{1}{2}\ln s + O(\frac{1}{s})\}.$$

To find the constant $\tilde{k}$ we use Euler's summation method and obtain the asymptotic expression for $\ln \tilde{f}(s)$ (see (B.6)-(B.8) of Appendix B). By comparing this expression with (5.2.2) we find that

$$(5.2.3) \qquad \tilde{k} = \frac{e^{\alpha}}{\sqrt{a}} \lim_{s\to\infty} \exp\{\frac{1}{2!}\cdot\int_{0}^{1}(2w-1) \sum_{i=1}^{\infty} \frac{\alpha s}{(w+i)\left[a(w+i)^{\alpha}+s\right]}\, dw\} .$$

Since the series in the integrand converges uniformly over $w \in [0,1]$, it can be integrated term by term (see Titchmarsh (1964, Section 1.71)). Therefore

$$\tilde{k} = \frac{e^{\alpha}}{\sqrt{a}} \lim_{s\to\infty} \exp\left\{\frac{\alpha}{2} \sum_{i=1}^{\infty} \int_{0}^{1} \frac{(2w-1)s\, dw}{(w+i)\left[a(w+i)^{\alpha}+s\right]}\right\}$$

Passing to the limit termwise (see Appendix I for justification) we obtain that

$$(5.2.4) \qquad \tilde{k} = \frac{e^{\alpha}}{\sqrt{a}} \exp\left\{\frac{\alpha}{2} \sum_{i=1}^{\infty} \int_{0}^{1} \frac{(2w-1)}{(w+i)}\, dw\right\} =$$

$$= \frac{e^{\alpha}}{\sqrt{a}} \exp\left\{\frac{\alpha}{2} \sum_{i=1}^{\infty} \left[2 - (2i+1)\ln(1+\tfrac{1}{i})\right]\right\} \quad .$$

There is no need to calculate the sum, it is sufficient to see that the sum exists and the exponent in (5.2.4) depends on α only. Since for $(\alpha=2, a=1)$ the quadratic case obtains for which we have $\tilde{k}=2\pi$ (see (5.1.23)), by substituting $(\alpha=2, a=1)$ into (5.2.4) we obtain

$$(5.2.5) \qquad \exp\left\{\sum_{i=1}^{\infty} \left[2-(2i+1)\ln(1+\tfrac{1}{i})\right]\right\} = \frac{2\pi}{e^{2}}$$

and the constant $\tilde{k}$ is equal to

$$(5.2.6) \quad \tilde{k} = \left(\frac{2\pi}{e^{2}}\right)^{\alpha/2} \cdot \frac{e^{\alpha}}{\sqrt{a}} = (2\pi)^{\alpha/2} \cdot a^{-1/2} \quad .$$

Thus, the distribution function F(t) is given by:

$$(5.2.7) \qquad F(t) = \frac{\tilde{k}}{2\pi i} \int_{d-i\infty}^{d+i\infty} c(s) \cdot s^{-1/2} e^{-\tau s^{1/\alpha}+st} ds,$$

where

$$(5.2.8) \qquad \tau = \frac{\pi a^{-1/\alpha}}{\sin(\pi/\alpha)} \quad ; \quad d > 0, \ \ln c(s) = O(1/s) \text{ as } s \to \infty .$$

The formula (5.2.7) is obtained using the inversion formula for the Laplace-Stieltjes transform (See Doetsch (1974)). In (5.2.7) the appropriate branch of $s^{1/\alpha}$ is that for which $s^{1/\alpha}$ is real and positive when s is positive. Now we must find an appropriate transform bringing the integral to the form suitable for saddle point analysis (See Appendix C). Let us try to make a substitution $s = zt^{\gamma}$. Then the exponent in the integral becomes

$$-\tau s^{1/\alpha} + st = -\tau z^{1/\alpha} t^{\gamma/\alpha} + zt^{\gamma+1} \quad .$$

To factorize the exponent let us chose γ satisfying $\gamma/\alpha = \gamma + 1$ i.e. let us take γ to be

$$\gamma = \frac{\alpha}{1-\alpha}$$

So, after performing the substitution $s = zt^{\alpha/(1-\alpha)}$ in (5.2.3) we obtain :

$$(5.2.9) \quad F(t) = \frac{\tilde{k}}{2\pi i} t^{\alpha/2(1-\alpha)} \int_{d(t)-i\infty}^{d(t)+i\infty} c(zt^{\frac{\alpha}{1-\alpha}}) z^{-1/2} \exp\{t^{\frac{1}{1-\alpha}}(-\tau z^{1/\alpha}+z)\} dz,$$

where $d(t) > 0$ and $d(t) \to 0$ as $t \to 0$. We see that $s \to \infty$ as $t \to 0$ since $\alpha > 1$.

Now the asymptotic expansion of $F(t)$ as $t \downarrow 0$ can be obtained by using the saddle point method (which is also known in the literature as "method of steepest descent" (See Murray (1974, p. 40-50)).

If we set

$$h(z) = -\tau z^{1/\alpha} + z,$$

the saddle point is found from the equation

$$(5.2.10) \qquad h'(z) = -\frac{\tau}{\alpha} z^{\frac{1}{\alpha} - 1} + 1 = 0$$

and, therefore, the only saddle point is

$$(5.2.11) \qquad z_o = \left(\frac{\alpha}{\tau}\right)^{\frac{\alpha}{1-\alpha}}$$

which is a point on the real axis.

Note that though $(\alpha/\tau)^{\alpha/(1-\alpha)}$ does not generally possess a unique value in the z-plane, only one of its values (lying on the real axis) satisfies (5.2.10) because of the particular choice of branch.

Since by (5.2.8) $c(s) = 1 + O(1/s)$ as $s \to \infty$, $c(s)$ will not affect the leading term of the expansion. To proceed further we must deform the path of integration in such a way that it passes through the saddle point and coincides in a neighborhood of the saddle point with the curve $\operatorname{Im} h(z) = 0$. Since

$$(5.2.12) \qquad h''(z_o) = \left(1 - \frac{1}{\alpha}\right)\left(\frac{\alpha}{\tau}\right)^{\alpha/(\alpha-1)} > 0 ,$$

z_o is a simple zero of $h'(z)$ and there are two lines satisfying $\operatorname{Im} h(z) = 0$ and passing through the point z_o. One of them is, of course, the real line but it is not suitable for our purposes since (by (5.2.12)) $\operatorname{Re}[h(z)]$ has a <u>minimum</u> along this direction, not a maximum. So we need to find the second one. Denoting its parametric equation by

$$(5.2.13) \qquad z(t) = \rho(t)e^{it} \quad , \text{ with } \rho(0) = z_o \quad , \quad -\pi < t < \pi,$$

the equation $\operatorname{Im} h(z) = 0$ becomes

$$(5.2.14) \qquad -\tau[\rho(t)]^{1/\alpha} \sin(t/\alpha) + \rho(t)\sin t = 0 \ ,$$

so the second curve has in polar coordinates the following form:

$$(5.2.15) \qquad \rho(t) = \left[\frac{\tau\sin(t/\alpha)}{\sin t} \right]^{\frac{\alpha}{\alpha-1}} \quad , \ -\pi < t < \pi \ .$$

We can verify immediately that $\rho'(0) = 0$. This means that the curve (5.2.15) forms an angle of $\pi/2$ with the real axis. It is known that along this second curve $\operatorname{Re}[h(z)]$ has a maximum, so it is the appropriate new integration path (See Fig. 5.2.1). Doing so and applying the saddle point method we obtain

$$(5.2.16) \quad F(t) \sim \frac{\tilde{k}}{2\pi i} \cdot 2t^{\alpha/2(1-\alpha)} [2\pi t^{-\frac{1}{1-\alpha}}]^{1/2} |h''(z_o)|^{-1/2} z_o^{-1/2} \exp[t^{\frac{1}{1-\alpha}} h(z_o) + \tfrac{i\pi}{2}]$$

as $t \downarrow 0$, or

$$(5.2.17)\quad F(t) \sim 2\tilde{k}\left(\frac{\alpha-1}{2\pi\alpha}\right)^{1/2} t^{-1/2} \exp\{(1-\alpha)\left(\frac{\alpha}{\tau}\right)^{\alpha/(1-\alpha)} t^{(1/(1-\alpha))}\}\ ,\ t \downarrow 0\ .$$

It is easy to verify that this form coincides with (5.1.18) when $\alpha = 2$. As well, we see that (5.2.17) has indeed the form predicted in (5.1.19).

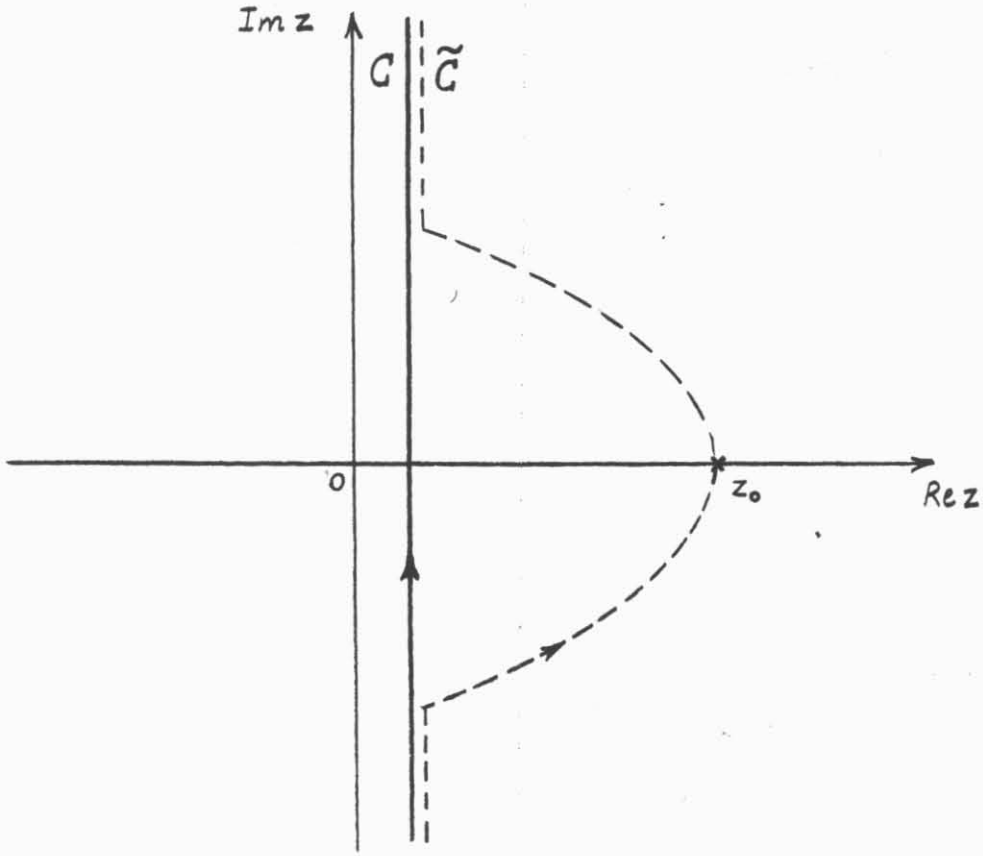


Fig. 5.2.1 Choice of a new integration path.

2. Finding the normalizing constants

Now we may find the appropriate normalizing constants since they are known to coincide with those of the main term given by (5.2.17). (See Theorem 3.1.6). We prove the following

Lemma 5.2.1 If the distribution function F satisfies the relation

$$F(t) \sim t^{\gamma} e^{-1/t^{\beta}} \quad ; \quad t \downarrow 0 \tag{5.2.18}$$

for $\beta > 0$, *then* F *belongs to the domain of attraction of* L, *the normalizing constants being*

$$\begin{cases} b_N = \frac{1}{\beta}(\ln N)^{-(\beta+1)/\beta} & \text{(scale constant)} \\ a_N = (\ln N)^{-1/\beta} + \dfrac{\gamma \ln(\ln N)}{\beta^2 (\ln N)^{(\beta+1)/\beta}} & \text{(shift constant)} . \end{cases} \tag{5.2.19}$$

Proof: The proof is straightforward. It is enough to show that

$$\lim_{N\to\infty} N(b_N t + a_N)^{\gamma} \exp\{- \frac{1}{(b_N t + a_N)^{\beta}}\} = \exp(t) \tag{5.2.20}$$

for every $t > 0$.

Instead of verifying (5.2.20) directly, we show, how the form (5.2.19) of b_N and a_N is obtained. Let us denote

$$F_1(t) = \exp(- \frac{1}{t^{\beta}}) \quad , \quad t > 0 . \tag{5.2.21}$$

We see that F_1 is a proper distribution function belonging to the domain of attraction of L. The normalizing constants $\tilde{a}_N$ and $\tilde{b}_N$ are found from the equations (3.1.15). Solving these equations we obtain, that

$$(5.2.22)\qquad \begin{cases} \tilde{a}_N = (\ln\ (N)^{-1/\beta} \\ \tilde{b}_N = (\ln N-1)^{-1/\beta} - (\ln N)^{-1/\beta} \sim \frac{1}{\beta}\ (\ln N)^{-(\beta+1)/\beta} \text{ as } N \to \infty. \end{cases}$$

Since $F(t)$ differs from $F_1(t)$ by a factor, varying regularly as $t \downarrow 0$, we guess and assume that the scale parameter b_N is the same for $F(t)$ and $F_1(t)$ and the normalizing constants a_N and b_N corresponding to F have the form

$$(5.2.23)\qquad \begin{cases} b_N = \frac{1}{\beta}\ (\ln N)^{-(\beta+1)/\beta} \\ a_N = (\ln N)^{-1/\beta} + \theta(\gamma,N,\beta). \end{cases}$$

We can show (See Appendix E for details) that (5.2.20) is satisfied when θ satisfies the equation

$$(5.2.24)\qquad \beta \cdot \theta \cdot (\ln N)^{(\beta+1)/\beta} = \frac{\gamma}{\beta} \ln(\ln N) :$$

that is, when

$$(5.2.25)\qquad \theta(\gamma,N,\beta) = \frac{\gamma\ \ln(\ln N)}{\beta^2 (\ln N)^{(\beta+1)/\beta}} .$$

This completes the proof.

□

We see that (5.2.20) is true also if N tends to infinity as a continuous variable. Therefore if

$$F_2(t) \sim kF(\tilde{a}t), \quad t \downarrow 0 \tag{5.2.26}$$

then F_2 belongs to the domain of attraction of L, the normalizing constants $\overline{a_N}$ and $\overline{b_N}$ being

$$\overline{a_N} = \frac{a_{Nk}}{\tilde{a}} \quad ; \quad \overline{b_N} = \frac{b_{Nk}}{\tilde{a}} \quad . \tag{5.2.27}$$

So, if

$$F_2(t) \sim kt^{\gamma}e^{-m/t^{\beta}} \ , \quad t \downarrow 0 \quad ; \quad k > 0 \ , \quad m > 0 \quad , \tag{5.2.28}$$

then the normalizing constants of $F_2(t)$ are

$$\begin{cases} \overline{a_N} = m^{1/\beta} a_{Nkm^{\gamma/\beta}} \\ \overline{b_N} = m^{1/\beta} b_{Nkm^{\gamma/\beta}} \end{cases} \tag{5.2.29}$$

And to obtain the normalizing constants for the F(t) of interest given by (5.2.17) we simply set, in (5.2.29),

$$\begin{cases} \gamma = -\frac{1}{2} \quad ; \quad m = (\alpha-1)\left(\frac{\alpha}{\tau}\right)^{\alpha/(1-\alpha)} ; \tau = \dfrac{\pi a^{-1/\alpha}}{\sin(\pi/\alpha)} \quad ; \\ \beta = \dfrac{1}{\alpha-1} \quad ; \quad k = 2\tilde{k}\left(\dfrac{\alpha-1}{2\pi\alpha}\right)^{1/2} \quad ; \tilde{k} = (2\pi)^{\alpha/2} \cdot a^{-1/2} \ . \end{cases} \tag{5.2.30}$$

Example. As an example let us calculate the normalizing constants for F(t) in the case of quadratic charge accumulation case, i.e. then $\lambda_j = (j+1)^2$, $j = 0,1,2,\ldots$. Let the number of cells be $N = 10^{10}$.

In this case we have

$$a = 1 \quad ; \alpha = 2 \; ; \gamma = -\frac{1}{2} \quad ; \quad m = \frac{\pi^2}{4} \quad ; \beta = 1 \quad ; \quad k = 2\sqrt{\pi}$$

Further,

$$m^{1/\beta} = \frac{\pi^2}{4} \quad ; \quad Nkm^{\gamma/\beta} = 10^{10} \times \frac{4}{\sqrt{\pi}} \quad ; \quad \ln(Nkm^{\gamma/\beta}) = 23.8398.$$

Therefore, by (5.2.19)

$$b_{Nkm^{\gamma/\beta}} = 0.001759 \quad ; \quad a_{Nkm^{\gamma/\beta}} = 0.039156$$

And finally, by (5.2.29) the normalizing constants for F(t) are:

$$\overline{b_N} = 0.004340 \quad ; \quad \overline{a_N} = 0.096612 \; .$$

Let us summarize the results by the means of following

Theorem 5.2.1 Let F(t) be the distribution function of infinite convolution of exponential densities with parameters $\lambda_j = a(j+1)^{\alpha}, j = 0,1,2,\ldots \; ; \alpha > 1$. Then F belongs to the domain of attraction of L, the normalizing constants being

$$(5.2.31)\qquad \begin{cases} b_N = \dfrac{\pi^\alpha}{a}\, p\, x^{-\alpha} \\[2ex] a_N = \dfrac{\pi^\alpha}{a}\, p\, x^{-\alpha}\left[\dfrac{x}{\alpha-1} - \dfrac{1}{2}(\alpha-1)\ln x\right] \end{cases}$$

where

$$(5.2.32)\qquad \begin{aligned} p &= \left[\frac{\alpha-1}{\alpha\sin(\pi/\alpha)}\right]^{\alpha} \\ x &= \ln N + \ln\{2^{\frac{\alpha+1}{2}}(\alpha-1)(\alpha\pi p)^{-1/2}\} \end{aligned}$$

The Fig. 5.2.2 illustrates the behaviour of $\{a_N\}$ and $\{b_N\}$ as functions of α and N.

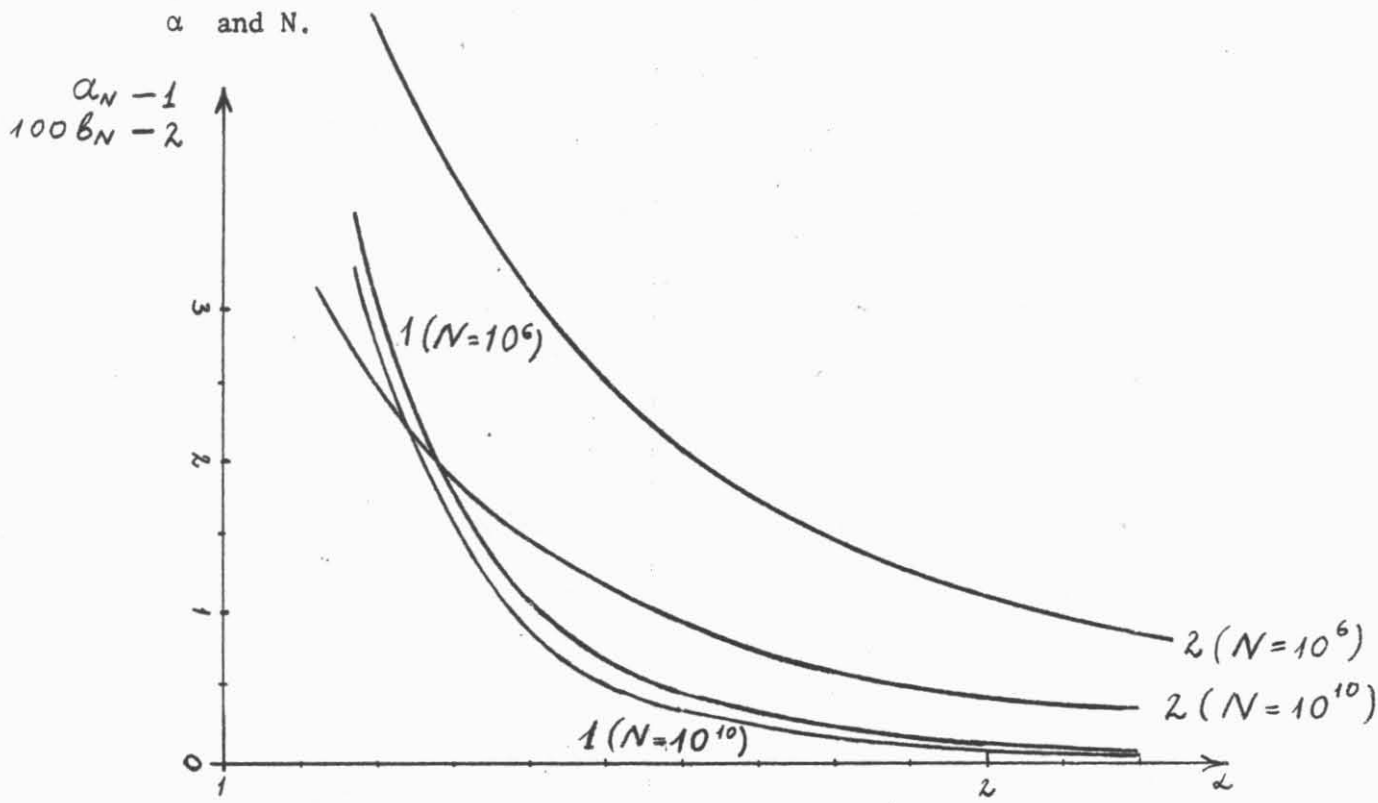


Fig. 5.2.2 Normalizing constants a_N (shift) and b_N (scales) for the power rate of charge accumulation case. Effect of varying α and N (a=1).

3. Relation to the general approach

In this subsection we shall use the general approach for the power charge accumulation rate case. (See paragraph 5.1.3). First of all, we have from (5.1.29), that

$$(5.2.33) \qquad F(\lambda(r)) \sim \frac{1}{\sqrt{2\pi}} \cdot \frac{\Lambda(r)}{r} \text{ as } r\to\infty ,$$

where $\Lambda(r)$ and $\lambda(r)$ are given by (5.1.26) and (5.1.24) respectively. Let us find the asymptotic representation for these functions as $r\to\infty$. We have (see 5.2.1):

$$(5.2.34) \qquad \lambda(r) = -\frac{d}{dr}\left[\ln \tilde{f}(r)\right] = \frac{\tau}{\alpha} r^{\frac{1}{\alpha}-1} - \frac{1}{2r} + O\left(\frac{1}{r^2}\right) \text{ as } r\to\infty ,$$

where τ is given by (5.2.8).

Since by definition of $\sigma(r)$,

$$(5.2.35) \qquad [\sigma(r)]^2 = -\lambda'(r),$$

let us differentiate the relation (5.2.34) and write

$$(5.2.36) \qquad \sigma(r) \sim \frac{\sqrt{\pi(\alpha-1)}}{\alpha} r^{\frac{1}{2\alpha}-1}, \ r\to\infty .$$

Since differentiation of asymptotic relations is in general invalid, (5.2.36) must be justified separately. For the purposes of the present subsection, let us suppose that (5.2.36) holds. The asymptotic behavior of $\tilde{f}(r)$ is given by (5.2.2).

Thus, $\Lambda(r)$ has the following representation,

$$(5.2.37) \qquad \Lambda(r) \sim \frac{e^{\frac{\tau}{\alpha} r^{1/\alpha} - \frac{1}{2}} \tilde{k} e^{-\tau r^{1/\alpha} + \frac{1}{2} \ln r}}{\sqrt{\tau(\alpha-1)}\,\alpha^{-1} r^{\frac{1}{2\alpha} - 1}} \quad , \quad r \to \infty \quad ,$$

So, the asymptotic expression for $F(\lambda(r))$ is

$$(5.2.38) \quad F(\lambda(r)) \sim \frac{\tilde{k}\alpha}{\sqrt{2\pi\tau e(\alpha-1)}} \; r^{\frac{1}{2}(1 - \frac{1}{\alpha})} \; e^{-\tau(1 - \frac{1}{\alpha}) r^{1/\alpha}} \quad , \quad r \to \infty$$

and to find the form of $F(t)$ as $t \to 0$ we must find the first terms of expansion of $\lambda^{-1}(t)$ for $t \downarrow 0$, i.e. we must expand the root of equation

$$(5.2.39) \qquad \sum_{j=1}^{\infty} \frac{1}{aj^{\alpha}+r} = t$$

in the neighborhood of t=0. This is a tedious task though it can be carried out using the asymptotic expansion of $\lambda(r)$ (See 5.2.34). For example, the first approximation gives

$$(5.2.40) \qquad \lambda^{-1}(t) \sim \left(\frac{t\alpha}{\tau}\right)^{\alpha/(1-\alpha)} \text{ as } t \downarrow 0$$

and substitution of this first approximation into (5.2.38) leads to the true expression for $F(t)$ except the constant of multiplication. To obtain this

constant we need the higher order terms in the representation (5.2.40) for $\lambda^{-1}(t)$.

The last paragraph illustrates the difficulties which arise when using the general approach. It seems that in many cases it is much easier to obtain the solution in a closed form using the saddle point analysis as was done in the previous paragraphs.

5.2.2 Geometric charge accumulation rates

In this section the case

$$\lambda_j = b\lambda^j \quad , \quad j = 0,1,2,\ldots \quad ; \lambda > 1$$

will be considered using the alternative approach. We shall see that this case is much more complicated than the previous one for three reasons.

a) The saddle point analysis for finding the form of $F(t)$ as $t \downarrow 0$ is much more refined,

b) There exist certain oscillations in the behaviour of $F(t)$ as $t \downarrow 0$.

c) The asymptotic form of $F(t)$ is not as convenient for finding the normalizing constants.

At the end of this section we shall consider a possible direct approach for finding the form of $F(t)$ as $t \downarrow 0$ using a differential equation which is satisfied by $F(t)$.

1. Asymptotic representation of $F(t)$ as $t \downarrow 0$

First of all let us denote

$$\psi(u) = \ln\left(\frac{\lambda^u}{\lambda^u + s}\right) . \tag{5.2.41}$$

Using Euler's summation method, we obtain:

$$\ln \tilde{f}(bs) = \sum_{i=o}^{\infty} \psi(i) = \int_o^{\infty} \psi(u)du + \frac{1}{2}[\psi(0) + \psi(\infty)] - R_1(s), \tag{5.2.42}$$

where

$$R_1(s) = -\frac{(\ln \lambda)^2}{2!} \int_o^1 (\omega^2 - \omega) \sum_{j=o}^{\infty} \frac{s\lambda^{\omega+j}}{(\lambda^{\omega+j}+s)^2} d\omega \tag{5.2.43}$$

We can see that $R_1(s)$ is bounded as $s \to \infty$. On the other hand, using the integration by parts technique we obtain

$$I = \int_o^{\infty} \psi(u)du = -\frac{1}{\ln\lambda} \int_o^{\infty} \frac{t}{1+\exp(t-\ln s)} dt . \tag{5.2.44}$$

(Here and in the sequel we denote by $\ln s$ the principal branch of the logarithm, i.e. the branch that takes real values for positive s).

But the last integral is the so called Fermi-Dirac integral and it has for large s the following representation (See Dingle (1973, p. 20)).

$$I = -\frac{1}{2 \ln \lambda} \{(\ln s)^2 + \frac{\pi^2}{3} + O(\frac{1}{s})\} \tag{5.2.45}$$

Thus, we have

$$(5.2.46) \qquad \tilde{f}(bs) = c(s) \cdot s^{-1/2} e^{-\frac{(\ln s)^2}{2 \ln \lambda}} ,$$

where

$$(5.2.47) \qquad c(s) = \left(\frac{s}{1+s}\right)^{1/2} \cdot \exp\left\{-\frac{\pi^2}{6 \ln\lambda} - R_1(s) + O\left(\frac{1}{s}\right)\right\}$$

We know that $c(s)$ is bounded as $s \to \infty$. Thus, by the inversion formula the distribution function F satisfies:

$$(5.2.48) \qquad F(t/b) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} c(s) \cdot s^{-3/2} \cdot e^{-\frac{(\ln s)^2}{2a} + st} ds,$$

where $a = \ln \lambda$ and $d > 0$.

After the change of variable $ts=x$ and performing the mapping $x = e^z$ (which is one-to-one in the strip $|\mathrm{Im}\, z| < \pi$), we obtain

$$(5.2.49) \qquad F(t/b) = \frac{(t)^{1/2}}{2\pi i} e^{-\frac{(\ln t)^2}{2a}} \int_C c\left(\frac{e^z}{t}\right) \cdot e^{-\frac{z^2}{2a} - \tilde{s}z + e^z} dz,$$

where $\tilde{s} = \frac{1}{2} - \frac{\ln t}{a}$.

The contour of integration C starts at $(+\infty - i\pi/2)$, crosses the real axis at the point $\ln(d \cdot t)$ and turns back to $(+\infty + i\pi/2)$. C lies entirely in the strip $|\mathrm{Im}\, z| < \pi/2$ and is symmetric with respect to the real axis. Completing the square under the exponent sign we obtain

$$(5.2.50) \qquad F(t/b) = \frac{1}{2\pi i} e^{a/8} \int_C c\left(\frac{e^z}{t}\right) e^{-\frac{(a\tilde{s}+z)^2}{2a} + e^z} dz$$

We are interested in an asymptotic expression for the integral in (5.2.50) as $\tilde{s} \to \infty$. In this form $F(t/b)$ is similar to the function $F_o(t)$ considered in de Bruijn (1953). The saddle point method applied there can be used here as well. The saddle point ζ of (5.2.50) satisfies

$$(5.2.51) \qquad \frac{d}{dz}\left[-\frac{(a\tilde{s}+z)^2}{2a} + e^z\right]\Bigg|_{z=\zeta} = 0$$

or,

$$(5.2.52) \qquad a\tilde{s} + \zeta = ae^{\zeta}$$

We see that for large $\tilde{s}$ there are two such points - one in the neighborhood of $\zeta_1 = -a\tilde{s}$ and the second in that of $\zeta_2 = \ln \tilde{s}$. We shall see that the main contribution comes from the second point. Let us denote this point by ζ and put $z = \omega+\zeta$ in (5.2.50). We obtain

$$(5.2.53) \qquad F(t/b) = \frac{1}{2\pi i} e^{\frac{a}{8} + \frac{2q-q^2}{2a}} \int_{C-\zeta} c\left(\frac{e^{\omega+\zeta}}{t}\right) e^{-\frac{\omega^2}{2a} + \frac{q}{a}(e^{\omega}-\omega-1)} d\omega ,$$

where $q = ae^{\zeta} = a\tilde{s} + \zeta$.

Transforming the contour appropriately and using the conformal mapping

$$(5.2.54) \qquad \xi = \{2(e^{\omega} - \omega - 1)\}^{1/2} ,$$

we obtain writing ω for $\omega(\xi)$)

$$(5.2.55) \qquad F(t/b) = \frac{1}{2\pi i} e^{\frac{a}{8} + \frac{2q-q^2}{2a}} \int_{-i\infty}^{i\infty} c\left(\frac{e^{\omega+\zeta}}{t}\right) e^{-\frac{\omega^2}{2a}} \frac{d\omega}{d\xi} \cdot e^{\frac{q\xi^2}{2a}} \, d\xi ,$$

(See Appendix G for details). We see now that the relevant saddle point is $\omega = 0$. After expanding the first part of the integrand

$$(5.2.56) \qquad c\left(\frac{e^{\omega+\zeta}}{t}\right) e^{-\frac{\omega^2}{2a}} \frac{d\omega}{d\xi} = a_o + \frac{a_1\xi}{1!} + \frac{a_2\xi^2}{2!} + \ldots$$

(here a_o will be equal to $c(\frac{e^{\zeta}}{t})$ and $a_1, a_2, \ldots$ are bounded functions of t), changing the variable $\xi = iz$ and integrating term by term we obtain the asymptotic expansion of $F(t/b)$ as $t \downarrow 0$:

$$(5.2.57) \qquad F(t/b) = \left(\frac{a}{2\pi q}\right)^{1/2} \cdot e^{\frac{a}{8} + \frac{2q-q^2}{2a}} \cdot \left\{ \sum_{k=o}^{m-1} (-1)^k a_{2k} \frac{(a/2q)^k}{k!} + O(|q|^{-m}) \right\}$$

Or, since the first term of the expansion is sufficient for our purposes, we have (note, that $ae^{\zeta} = q$ by definition)

$$(5.2.58) \qquad F(t/b) \sim c\left(\frac{q}{at}\right) \cdot \left(\frac{a}{2\pi q}\right)^{1/2} e^{\frac{a}{8} + \frac{2q-q^2}{2a}} , \qquad t \downarrow 0.$$

We see that q can be localized by the following procedure. By (5.6.52), q satisfies the equation

$$(5.2.59) \qquad p + \ln q = q \quad ,$$

where $p = a\tilde{s} - \ln a = -\ln t + \frac{a}{2} - \ln a$. Let us take the first approximation

$$(5.2.60) \qquad q_1 = p + \varepsilon_1$$

Then ε_1 satisfies

$$(5.2.61) \qquad \ln p + \ln(1 + \frac{\varepsilon_1}{p}) = \varepsilon_1$$

From the last equation we find that $\varepsilon_1 = O(\ln p)$ (Assuming that this is not true leads immediately to a contradiction). So, the first approximation for q is

$$(5.2.62) \qquad q_1 = p + O(\ln p) \ .$$

Now, substituting q_1 into the original equation (5.2.59) we obtain the second approximation for q:

$$(5.2.63) \qquad q_2 = p + \ln(p + O(\ln p)) = p + \ln p + O(\frac{\ln p}{p})$$

The third approximation is obtained in the same way :

$$(5.2.64) \qquad q_3 = p + \ln(p + \ln p + O(\frac{\ln p}{p})) =$$

$$= p + \ln p + \ln(1 + \frac{\ln p}{p} + O(\frac{\ln p}{p^2})) =$$

$$= p + \ln p + [\frac{\ln p}{p} + O(\frac{\ln p}{p^2})] - \frac{1}{2}[\frac{\ln p}{p} + O(\frac{\ln p}{p^2})]^2 + O[(\frac{\ln p}{p})^3] =$$

$$= p + \ln p + \frac{\ln p}{p} - \frac{(\ln p)^2}{2p^2} + O(\frac{\ln p}{p^2})$$

We can continue this procedure to obtain higher order terms. Possibly, it may be shown that the resulting series will be convergent rather than asymptotic (See de Bruijn (1970, p. 25) for a similar case). Using q_3 as an approximation for q we can express the asymptotic representation (5.2.58) in terms of t. The expression is, however, too complicated to deal with and therefore we shall use another way to find the normalizing constants.

2. <u>Oscillating behaviour of F(t) as $t \downarrow 0$</u>

We know that as $t \downarrow 0$ (and consequently $q/at \to \infty$) the function $c(q/at)$ is bounded and positive. But, in contrast to the case of power charge accumulation rates, this function oscillates and does not approach a limit as $t \downarrow 0$. Since by (5.2.59)

$$(5.2.65) \qquad \ln(\frac{q}{at}) = q - \frac{a}{2} \quad ,$$

using (5.2.47) we obtain that as $t \downarrow 0$

$$(5.2.66)\quad c(\frac{q}{dt}) \sim \exp(-\frac{\pi^2}{6a})\exp\{\frac{a^2}{2}\int_o^1(\omega^2-\omega)\sum_{j=o}^{\infty}\frac{\exp[a\omega+aj-\ln(q/at)]}{\{1+\exp[a\omega+aj-\ln(q/at)]\}^2}\,d\omega\}$$

or, after using (5.2.65) and changing the summation index,

$$(5.2.67)\quad c(\frac{q}{at}) \sim \exp(-\frac{\pi^2}{6a})\cdot\exp\{\frac{a^2}{2}\int_o^1(\omega^2-\omega)\sum_{j=-\infty}^{\infty}\frac{\exp[a(\omega+j+\alpha)]}{\{1+\exp[a(\omega+j+\alpha)]\}^2}\,d\omega\},$$

where $-1/2 < \alpha < 1/2$ is 1/2 minus the fractional part of q/a. It can be shown (See Appendix H) that the right hand side of (5.2.67) does depend on α though the dependence is weak. So, the function $c(\frac{q}{at})$ oscillates as $t \downarrow o$ and its value at the point t is determined meainly by the fractional part of (q/a).

Thus, finally the representation of $F(t)$ as $t \downarrow 0$ is :

$$(5.2.68)\qquad F(t/b) \sim k\exp\{-\frac{q^2-2q}{2a}-\frac{1}{2}\ln q+\phi(q)\},$$

where

$$(5.2.69)\qquad k = (\frac{a}{2\pi})^{1/2}\exp(\frac{a}{8}-\frac{\pi^2}{6a})$$

and $\phi(q)$ is the second exponent on the right hand side of (5.2.66) (or (5.2.67)). It can be easily verified that $\phi'(q)$ is uniformly bounded and continuous as $q > 0$. We shall use the form (5.2.68) of $F(t)$ for finding the expressions for the normalizing constants.

3. Main equation for finding the quantiles of $F(t)$ as $t \downarrow 0$

After the asymptotic form of $F(t)$ has been found, we can deal with the problem of finding expressions for the normalizing constants. Since the representation (5.2.68) depends on q only and by (5.2.65)

$$t = \frac{q}{a} e^{-q + \frac{a}{2}} , \tag{5.2.70}$$

in order to determine the normalizing constants it is sufficient to localize the quantiles of $F(t)$ for small values of t (See Corollary 3.1.4): that is, it is sufficient to localize the solution of the equation

$$q^2 - 2q + a \ln q - 2a\, \phi(q) = x \tag{5.2.71}$$

when $x \to \infty$. We shall proceed by the method of iterations. Let us find the first approximation. Evidently, the solution may be represented as

$$q(x) = 1 + (1+x)^{1/2} + \varepsilon(x), \tag{5.2.72}$$

where $\varepsilon(x) \to 0$ as $x \to \infty$. So, the first iteration will be

$$q_1(x) = x^{1/2} + O(1) \tag{5.2.73}$$

Now let us put the equation (5.2.71) into the form

$$q = [x + 2q - a \ln q + 2a\phi(q)]^{1/2} \tag{5.2.74}$$

Substituting (5.2.73) into the right hand side of (5.2.74) we obtain the second approximation.

$$(5.2.75)\quad q_2(x) = x^{1/2}\{1 + \frac{1}{x}[2(x^{1/2}+O(1)) - a\ln(x^{1/2}+O(1)) + 2a\phi(x^{1/2}+O(1))]\}^{1/2}$$

Or, using the expansions

$$(5.2.76)\qquad (1+t)^{1/2} = 1 + \frac{t}{2} - \frac{t^2}{8} + \ldots \quad ; \quad \ln(1+t) = t - \frac{t^2}{2} + \ldots ,$$

the second approximation is

$$(5.2.77)\qquad q_2(x) = x^{1/2} + 1 - \frac{a}{4}\cdot x^{-1/2}\ln x + O(x^{-1/2}).$$

Substituting $q_2(x)$ into the right hand side of (5.2.74), we find the third approximation

$$(5.2.78)\qquad q_3(x) = x^{1/2} + 1 - \frac{a}{4}\cdot x^{-1/2}\ln x +$$

$$+ x^{-1/2}[\frac{1}{2} + a\phi(x^{1/2} + 1 - \frac{a}{4}x^{-1/2}\ln x)] + O(\frac{1}{x})$$

because (by boundness of ϕ')

$$(5.2.79)\qquad \phi(x^{1/2} + 1 - \frac{a}{4}x^{-1/2}\ln x + O(x^{-1/2})) = \phi(x^{1/2} + 1 - \frac{a}{4}x^{-1/2}\ln x) +$$

$$+ O(x^{-1/2})$$

This iterative procedure can be continued to obtain more precise approximations.

4. Finding the normalizing constants

Since the normalizing constants a_N (shift) and b_N (scale) can be found from the equations (3.1.15), by using (5.2.68), (5.2.69) and (5.2.77) we obtain

$$(5.2.80)\quad \begin{cases} a_N = \frac{1}{ab}\cdot e^{a/2}\cdot\{x_1^{1/2}+1 - \frac{a}{4}x_1^{-1/2}\ln x_1+O(x_1^{-1/2})\}e^{-\{x_1^{1/2}+1 - \frac{a}{4}x_1^{-1/2}\ln x_1+O(x_1^{-1/2})\}} \\ a_N+b_N=\frac{1}{ab}e^{a/2}\{x_2^{1/2}+1-\frac{a}{4}\cdot x_2^{-1/2}\ln x_2+O(x_2^{-1/2})\}e^{-\{x_2^{1/2}+1-\frac{a}{4}x_2^{-1/2}\ln x_2+O(x_2^{-1/2})\}}, \end{cases}$$

where

$$(5.2.81)\qquad x_1 = 2a\ \ln(kN) \quad ; \quad x_2 = 2a[\ln(kN) - 1].$$

From the second equation of (5.2.80) we obtain

$$(5.2.82)\quad a_N+b_N = \frac{1}{ab}\cdot e^{a/2}\{x_1^{1/2}+1 - \frac{a}{4}x_1^{-1/2}\ln x_1+O(x_1^{-1/2})\}\ e^{-\{x_1^{1/2}+1-\frac{a}{4}\cdot x_1^{-1/2}\ln x_1+O(x_1^{-1/2})\}}$$

$$\times\ \{1 + ax_1^{-1/2} + O(\frac{1}{x_1})\}\quad ,$$

so, using the Theorem 3.1.5, we can choose the constant b_N as

$$(5.2.83)\qquad b_N = \frac{1}{b}\ e^{a/2}\cdot e^{-\{x_1^{1/2} + 1\}}\quad .$$

The same theorem says that a_N can be chosen as

$$(5.2.84) \qquad a_N = \frac{1}{ab}\, e^{a/2}\cdot\{x_1^{1/2}+1\}\cdot e^{-\{x_1^{1/2}+1 - \frac{a}{4}x_1^{-1/2}\ln x_1 + O(x_1^{-1/2})\}}$$

We can see that the term $O(x_1^{-1/2})$ cannot be omitted.

This is not surprising as it reflects the oscillating behaviour of the distribution function at the origin. But if we specify this term by the means of (5.2.78) (i.e. if we take a more precise estimate of q), the remaining term $O(1/x_1)$ can be omitted. In summary, we have proved the following:

<u>Theorem 5.6.1</u> Let F be the distribution function of an infinite convolution of exponential densities with parameters $\lambda_j = b\lambda^j$; $j = 0,1,2,\ldots$; $\lambda > 1$. Then F belongs to the domain of attraction of L, the normalizing constants being

$$(5.2.85) \quad \begin{cases} b_N = \frac{1}{b}\exp\{\frac{a}{2} - 1 - x^{1/2}\} \\ a_N = \frac{1}{ab}(x^{1/2}+1)\exp\{\frac{a}{2} - [x^{1/2} + 1 - \frac{a}{4}x^{-1/2}\ln x + \\ \quad + x^{-1/2}[\frac{1}{2} + a\phi(x^{1/2} + 1 - \frac{a}{4}x^{-1/2}\ln x)]]\} \end{cases}$$

where

$$(5.2.86) \qquad a = \ln\lambda \;;\quad x = 2a\ln(kN) \;;\quad k = (\frac{a}{2\pi})^{1/2}\exp(\frac{a}{8} - \frac{\pi^2}{6a}) \quad \text{and}$$

$\phi(q)$ is given by

$$(5.2.87)\quad \phi(q) = \frac{a^2}{2}\int_o^1 (\omega^2-\omega)\sum_{j=o}^{\infty} \frac{\exp[a\omega+aj+\frac{a}{2}-q]}{\{1+\exp[a\omega+aj+\frac{a}{2}-q]\}^2}\, d\omega$$

Remark The form (5.2.87) is rather inconvenient for computing $\phi(q)$. We can simplify this form by using an integration by parts technique or (what is essentially the same) simply by using the equality (5.2.42). Indeed, by (5.2.42)

$$(5.2.88)\quad \phi(q) = -R_1(s)\big|_{s=e^{q-\frac{a}{2}}} = \sum_{i=o}^{\infty} \ln\left(\frac{e^{ai}}{e^{ai}+e^{q-\frac{a}{2}}}\right) + \frac{1}{2}\ln(1+e^{q-\frac{a}{2}}) +$$

$$+ \frac{1}{a}\int_o^{\infty} \frac{tdt}{1+\exp(t-(q-\frac{a}{2}))} =$$

$$= -\sum_{i=1}^{\infty} \ln(1+e^{q-ai-\frac{a}{2}}) - \frac{1}{2}\ln(1+e^{q-\frac{a}{2}}) +$$

$$+ \frac{1}{2a}(q-\frac{a}{2})^2 + \frac{\pi^2}{6a} - \frac{1}{a}\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i^2} e^{-i(q-\frac{a}{2})}.$$

after the complete expansion for the Fermi-Dirac integral has been used (See Dingle (1973, p. 20)).

Example Let us calculate the normalizing constants for F(t) in the case that the rates of birth are given by $\lambda_j = (1.2)^j$, $j = 0,1,2,\ldots$ and $N = 10^{10}$.

Using (5.2.86), (5.2.88) and (5.2.85) we find

$$b_N = 0.048660 \quad ; \quad a_N = 0.678435 \ .$$

Behaviour of the normalizing sequences as functions of λ and N is illustrated by Fig. 5.2.3.

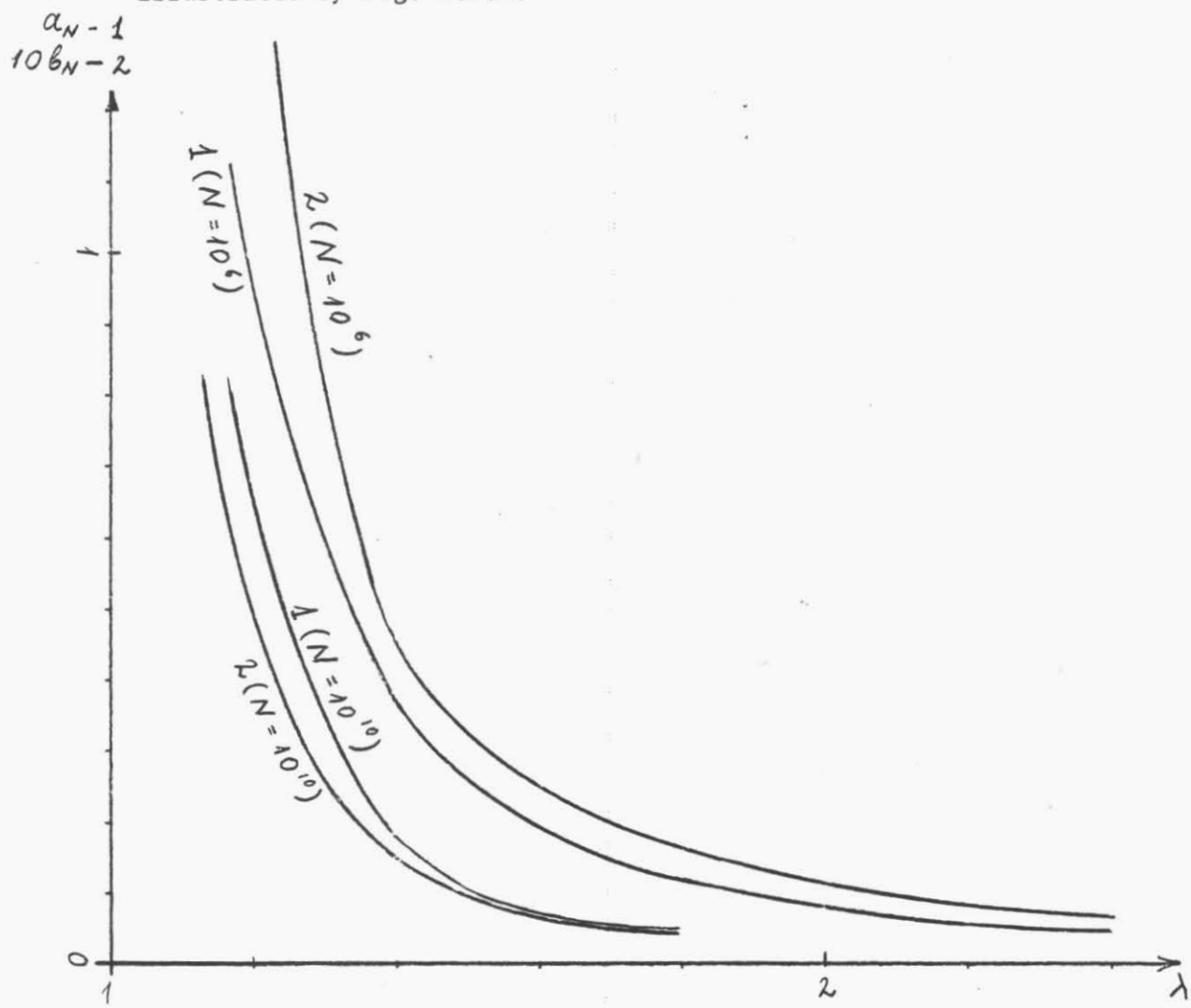


Fig. 5.2.3 Normalizing constants for the case $\lambda_j = \lambda^j$.

5.2.3 The direct approach in the geometric rates case

In this paragraph we show that the asymptotic properties of F at $t \downarrow 0$ (geometric rates case) may be obtained using the differential equation which is satisfied by F. We shall see that this equation is similar to that considered in De Bruijn's work (1953) in connection with Mahler's partition problem in number theory. This explains, why the integrals appearing in our investigation of this case (See, for example, (5.2.50)) are similar to those appearing in his work.

Let us make a non-restricting assumption that $\{\lambda_i = \lambda^i, \quad i = 0,1,\ldots\}$.

Since the Laplace transform $\tilde{F}$ of F is given by

$$(5.2.89) \qquad \tilde{F}(s) = \frac{1}{s} \prod_{i=0}^{\infty} \frac{\lambda^i}{\lambda^i + s} \quad ,$$

$\tilde{F}$ satisfies the equation

$$(5.2.90) \qquad \tilde{F}(s) = \lambda(1 + \lambda s)\tilde{F}(\lambda s).$$

Inverting both sides of the last equation, we obtain

$$(5.2.91) \qquad F(t) = F(\tfrac{t}{\lambda}) + F'(\tfrac{t}{\lambda}) \; .$$

Writing $\lambda = e^a$, $t = \lambda e^{-ax}$, $G(x) = F(e^{-ax})$ we have the relation

$$(5.2.92) \qquad G'(x) = ae^{-ax}[G(x) - G(x-1)]$$

and are interested in the behaviour of $G(x)$ as $x \to \infty$.

Furthermore, on considering (5.2.91), two or more of the terms in (5.2.91) must have the same order of magnitude as $t \downarrow 0$. However, from (5.2.89) and a Tauberian theorem (See Feller (1971, p. 443)) we know that, for $\lambda > 1$,

$$\frac{F(t/\lambda)}{F(t)} \to 0 \quad \text{as} \quad t \downarrow 0 \tag{5.2.93}$$

Thus we may conclude that, as $x \to \infty$, the solution of (5.2.92) behaves as that of

$$G'(x) = -ae^{-ax}G(x-1), \tag{5.2.94}$$

which is De Bruijn's equation except for the sign of a in the exponent. Indeed, solving the last equation we obtain a true asymptotic representation of $F(t)$ as $t \downarrow 0$, except for the oscillating factor which was lost when passing from (5.2.92) to (5.2.94). Therefore, to obtain the exact form of asymptotic behaviour of $F(t)$, we must deal with (5.2.92) which is a much more complicated equation than that considered by De Bruijn. Solution of (5.2.94) may serve as a first approximation.

5.3 Inference for the pure birth model

In this section we apply the methods described in Section 3.3 to estimate the parameters of the infinite charge breakdown models and to obtain the form of interconnections between the distribution of the interbreakdown time and the parameters of the model. As the results of this chapter show, $F(t)$, the d.f. of the time to explosion in a single cell, belongs to the domain of

attraction of L. So, the process of successive breakdowns is either a renewal process in the regenerative case (with time between renewals which is a linear transform of the double exponential random variable) or a non-homogeneous Poisson process (non regenerative case). We consider both types of models.

5.3.1 Regenerative case

We know, that in this case the interbreakdown time T is given by $(a_N + b_N \cdot X)$, where X is the random variable distributed in accordance with the d.f. L(t). It can be easily verified (See David (1981, p.260)) that the mean E and the variance V^2 of X are given by:

$$E = -C = -0.577 \quad ; \quad V^2 = \sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad . \tag{5.3.1}$$

First of all, let us see what influence the area of an insulator has on the interbreakdown time distribution. Suppose that the number of cells changes from N to N_1 (in other words, the area of insulating material increases N_1/N times). Denote by T^1 the interbreakdown time corresponding to N_1 cells. Then

$$\begin{cases} \dfrac{ET}{ET^1} = \dfrac{a_N + b_N E}{a_{N_1} + b_{N_1} E} = \dfrac{a_N}{a_{N_1}} \cdot \dfrac{1 + (b_N/a_N)E}{1 + (b_{N_1}/a_{N_1})E} \\[2ex] \dfrac{\sigma_T}{\sigma_{T^1}} = \dfrac{b_N}{b_{N_1}} \end{cases} \tag{5.3.2}$$

But since in the present model (b_N/a_N) tends to zero as $N \to \infty$ (See Corollary 3.1.3) the first relation in (5.3.2) becomes

$$\frac{ET}{ET^1} \approx \frac{a_N}{a_{N_1}} \tag{5.3.3}$$

Let us give some rough estimations for both pure birth charge accumulation models.

(i) Geometric charge accumulation rates

Notice that since the constant b is a factor in the normalizing constants (and therefore, in the mean and standard deviation of the interbreakdown time T) all the expressions containing ratios of these constants only, will not depend on b.

As estimate of σ_T will therefore be

$$\hat{\sigma}_T = V \cdot \frac{1}{b} \exp(\frac{a}{2} - 1) \exp\{-(2a \ln(kN))^{1/2}\} \approx V \cdot \frac{1}{b} \exp(\frac{a}{2} - 1) \exp\{-(2a \ln N)^{1/2}\} , \tag{5.3.4}$$

and for ET

$$\hat{ET} = a_N + b_N E \approx a_N \approx \frac{1}{ab} \exp(\frac{a}{2} - 1) \cdot (2a \ln N)^{1/2} \exp\{ -(2a \ln N)^{1/2}\} \tag{5.3.5}$$

These formulas show what kind of dependence between the interbreakdown time distribution and the area of insulator can be expected should the regenerative case be appropriate. As well, they show the kind of dependence between the interbreakdown time distribution and other parameters of the model when the area of insulator remains unchanged.

The parameters a, b and N cannot, of course, be estimated from the set of observations, corresponding to the same experimental conditions (See paragraph 3.3). Hence, to perform the estimation we need to take an extra sample, corresponding to a new situation, to equate all the sets of estimates with their analytic expressions (given by (5.2.85) and solve for the parameters.

We shall not display the relevant equations. Instead of doing so, let us demonstrate, how an additional sample, corresponding to a slight change in area may be used to obtain rough estimates for a, b and N.

Suppose that the area of insulator changes from S to $S_1 = \rho S$, where ρ is close to 1. (That is the number of cells changes from N to $N_1 = \rho N$). Denoting by T and T^1 the interbreakdown times corresponding to N and N_1 respectively, we obtain

$$(5.3.6) \quad ET^1 \approx \frac{a_{N_1}}{a_N} ET \approx \frac{(x + 2a \ln \rho)^{1/2} \exp\{-(x + 2a \ln \rho)^{1/2}\}}{x^{1/2} \exp\{-x^{1/2}\}} ET$$

and, expanding the numerator at the point x we have

$$(5.3.7) \qquad ET^1 \approx [1 - ax^{-1/2}(1-x^{-1/2})(\ln \rho)]ET$$

In the same way we obtain

$$(5.3.8) \qquad \sigma_{T^1} \approx \frac{b_{N_1}}{b_N} \sigma_T = \frac{\exp\{-(x + 2a \ln \rho)^{1/2}\}}{\exp\{-x^{1/2}\}} \sigma_T \approx$$

$$\approx [1 - ax^{-1/2}(\ln \rho)]\sigma_T$$

These results show that the relative change in ET and σ_T is proportional to $\ln \rho$. The relations (5.3.7) and (5.3.8) enable one to find estimators $\hat{a}$, $\hat{N}$ and $\hat{b}$ for the parameters of the model as soon as two sets of interevent times $\{T_j,\ j = 1,2,\ldots,n\}$ and $\{T_i^1,\ i = 1,2,\ldots,m\}$ (corresponding to areas S and $S_1 = \rho S$ of insulator respectively) are given. If $(\overline{T}, S_T)$ and $(\overline{T}^1, S_{T^1})$ are sample mean and sample standard deviation of both sets of data, (5.3.7) and (5.3.8) provide the following equations for estimators $\hat{a}$ and $\hat{N}$:

$$\begin{cases} ax^{-1/2}(1-x^{-1/2})(\ln \rho) = 1 - (\overline{T}^1/T) & (5.3.9) \\ ax^{-1/2}(\ln \rho) = 1 - (S_{T_1}/S_T) \ , & (5.3.10) \end{cases}$$

where x is given by (5.2.85). Dividing these equations one by another we obtain

$$\hat{x} = \left[\frac{1-(S_{T_1}/S_T)}{(\overline{T}_1/T)-(S_{T_1}/S_T)} \right]^2 \qquad (5.3.11)$$

So, the estimator of a is given by

$$\hat{a} = \frac{[1 - (S_{T_1}/S_T)]^2}{(\overline{T}_1/T) - (S_{T_1}/S_T)} \cdot (\ln \rho)^{-1} \qquad (5.3.12)$$

and the estimator of N is

$$(5.3.13) \qquad \hat{N} = \exp\{\frac{\hat{3}x + \pi^2}{6\hat{a}} - \frac{1}{2}\ln(\frac{\hat{a}}{2\pi}) - \frac{\hat{a}}{8}\} \quad ,$$

where $\hat{x}$ and $\hat{a}$ are given by the previous two formulae.

To estimate b we could use the relation $\sigma_T = b_N V$. However, this approach would not make use of all the information of hand, so some pooled estimator for b would be preferable. Such an estimator can be obtained using the fact that random variables

$$\frac{T_j - a_N}{b_N}, \quad j = 1,2,\ldots,n \quad , \quad \text{and} \quad \frac{T_i^1 - a_{N_1}}{b_{N_1}} \quad , \quad i = 1,2,\ldots,m$$

can be considered as n+m independent variables with the same distribution function L. Thus, as m and n are large, the estimator for b can be found from the equation

$$(5.3.14) \qquad \frac{n\,\dfrac{\overline{T}-a_N}{b_N} + m\,\dfrac{\overline{T}^1-a_{N_1}}{b_{N_1}}}{m+n} = E$$

But (by 5.2.84)

$$(5.3.15) \qquad \frac{a_N}{b_N} \approx \frac{1}{\hat{a}}\,\hat{x}^{1/2} \quad ; \quad \frac{a_{N_1}}{b_{N_1}} \approx \frac{1}{\hat{a}}\,(\hat{x} + 2\hat{a}\ln\rho)^{1/2}$$

Further,

$$(5.3.16) \quad b_N \approx \frac{1}{b}\exp\{\frac{\hat{a}}{2} - 1 - \hat{x}^{1/2}\} \; ; \; b_{N_1} \approx \frac{1}{b}\exp\{\frac{\hat{a}}{2} - 1 - (\hat{x} + 2\,\hat{a}\ln\rho)^{1/2}\}$$

So, the estimator $\hat{b}$ is

$$(5.3.17)\quad \hat{b} = \frac{E\cdot(m+n) + \frac{n}{\hat{a}}\hat{x}^{1/2} + \frac{m}{\hat{a}}(\hat{x} + 2\hat{a}\ln\rho)^{1/2}}{[n\overline{T}\exp(\hat{x}^{1/2}) + m\overline{T}^{1}\exp((\hat{x} + 2\hat{a}\ln\rho)^{1/2})]\exp[-\frac{\hat{a}}{2} - 1]} .$$

Finally, when changing the experimental conditions in order to take an additional sample, the possible changes include change of the voltage, change of thickness of the insulating material, change of the electrode material, change of the polarity, change of the area of the insulator etc. The last possibility seems the most practical one because the area of insulator is connected directly with the number of cells N. However, it is not the best one from a statistical point of view. The reason for this will be explained in subsection (iii).

(ii) Power charge accumulation rates

In this case the normalizing constants are expressed by (5.2.31). The constant a acts as a constant multiplier. (Like the constant b in the previous case). The mean and standard deviation of the interbreakdown time depend on N via

$$(5.3.18)\qquad \sigma_T = \frac{\pi^{\alpha}p}{a}\cdot V\cdot x^{-\alpha}$$

$$(5.3.19)\qquad ET = \frac{\pi^{\alpha}p}{a}x^{-\alpha}\left[\frac{x}{\alpha-1} - \frac{1}{2}(\alpha-1)\ln x + E\right],$$

where p and x are given by (5.2.32). Since $x = \ln N + \text{const}$, the last formulas show the kind of dependence between area of insulator and distribution of interbreakdown times.

If the number of cells changes from N to $N_1 = \rho N$ $(\rho \approx 1)$, T and T^1 being (respectively) the interbreakdown times, then

$$(5.3.20)\qquad \begin{cases} ET^1 = \dfrac{a_{N_1} + b_{N_1} E}{a_{N_2} + b_{N_2} E} \cdot ET \\[2ex] \sigma_{T_1} = \dfrac{b_{N_1}}{b_N}\ \sigma_T \end{cases}$$

Acting as in previous case, we obtain

$$(5.3.21)\qquad \begin{cases} ET^1 \approx \left[1 - \dfrac{\alpha}{x} + \dfrac{\left(\dfrac{1}{\alpha-1} - \dfrac{\alpha-1}{2x}\right)\ln \rho}{\dfrac{x}{\alpha-1} - \dfrac{\alpha-1}{2}\ln x + E}\right] ET \\[3ex] \sigma_{T_1} \approx \left[1 - \dfrac{\alpha}{x}\ln \rho\right] \sigma_T \end{cases}$$

Using (5.3.21) we can find the estimators $\hat{N}$ and $\hat{\alpha}$ as soon as two sets of data (corresponding to various areas) are at hand. (See the previous case). An estimator $\hat{a}$ is obtained as previously, using (5.3.14).

$$(5.3.22)\quad \hat{a} = \frac{E\cdot(m+n) + n[\frac{\hat{x}}{\hat{\alpha}-1} - \frac{\hat{\alpha}-1}{2}\ln\hat{x}] + m[\frac{\hat{x}+\ln\rho}{\hat{\alpha}-1} - \frac{\hat{\alpha}-1}{2}\ln(\hat{x}+\ln\rho)]}{[n\overline{T}\hat{x}^{\hat{\alpha}} + m\overline{T}^{1}(\hat{x}+\ln\rho)^{\hat{\alpha}}]\cdot[\frac{\hat{\alpha}\sin(\pi/\hat{\alpha})}{\pi(\hat{\alpha}-1)}]^{\hat{\alpha}}} .$$

(iii) <u>Remark</u>

Let $F^{-1}(\cdot)$ be the inverse function of the d.f. F. It is known (See Gnedenko (1943), De Haan (1971, p. 76) that F belongs to the domain of attraction of L with normalizing constants a_N and b_N if and only if the left tail quantiles $F^{-1}(c/N)$ satisfy

$$(5.3.23)\qquad \frac{F^{-1}(c/N) - a_N}{b_N} \to \ln c \qquad \text{as } N \to \infty .$$

The last relation may be used to estimate the left tail quantiles of F (after a_N and b_N have been estimated). Suppose the number of cells is changed to $N_1 = \rho N$. Then the new normalizing constants are

$$(5.3.24)\qquad a_{N_1} = F^{-1}(\frac{1}{\rho N}) ,$$

$$(5.3.25)\qquad b_{N_1} = F^{-1}(\frac{e}{\rho N}) - F^{-1}(\frac{1}{\rho N}) ,$$

satisfying (by 5.3.23)

$$(5.3.26)\qquad \frac{a_{N_1}-a_N}{b_N} \to -\ln\rho,\ \frac{b_{N_1}}{b_N} \to 1 \quad \text{as } N \to \infty .$$

So, the estimators $\hat{a}_N$ and $\hat{b}_N$ carry information on the normalizing constants a_{N_1}, b_{N_1} for any number of cells N_1 as N tends to infinity. This means that changing the area of the insulator is not the best way to estimate the internal parameters of the model (other changes of experimental conditions, mentioned in the end of subsection (i) might be preferable). Nevertheless, such estimation is possible in principle since the number of cells is finite and analytic expressions for the normalizing constants provide much stronger relations than (5.3.26). Let us take, for example, the case $F(t) = \exp(-(\alpha t)^{-\beta})$ $,t,\alpha,\beta > 0$. Then, by Lemma 5.2.1 the normalizing constants are

$$(5.3.27)\qquad \begin{cases} a_N = \dfrac{1}{\alpha}\,(\ln N)^{-1/\beta} \\[2ex] b_N = \dfrac{1}{\alpha\beta}\,(\ln N)^{-\frac{\beta+1}{\beta}} \end{cases}$$

and we see, that

$$(5.3.28)\qquad \begin{cases} a_{N_1} - a_N = b_N\left[-\ln\rho + \dfrac{1}{2!}\left(\dfrac{1}{\beta}+1\right)\dfrac{(\ln\rho)^2}{\ln N} + O\left(\dfrac{1}{(\ln N)^2}\right)\right] \\[2ex] \dfrac{b_{N_1}}{b_N} = 1 - \left(1+\dfrac{1}{\beta}\right)\dfrac{\ln\rho}{\ln N} + O\left(\dfrac{1}{(\ln N)^2}\right) \end{cases}$$

The last relations tell us much more than (5.3.26) and can be used (together with (5.3.27) to estimate α , β and N if the estimators $\hat{a}_{N_1}$,

a_N, b_{N_1}, b_N are at hand. This remark shows why the estimation procedure, applied is subsections (i) and (ii) is legitimate.

5.3.2 The non-regenerative case

In this section we consider the non-regenerative model with zero initial and pure birth charge accumulation process with geometric rates (the case of power rates may be treated analogously). Suppose we observe the sequence of the first k breakdowns $\{\xi_{(1)},\ldots,\xi_{(k)}\}$. Then the normalizing constants a_N and b_N may be estimated using the formulae (3.3.13)-(3.3.14). In the case that one of the three parameters (λ,b,N) of the model is known, the other two may be estimated by solving the equations

$$(5.3.29) \qquad \begin{cases} a_N = \hat{a}_N \\ b_N = \hat{b}_N \end{cases},$$

where expressions for a_N and b_N are given by (5.2.85).

In the case that all the three parameters (λ,b,N) are not known, to estimate them we can take the samples of the k first breakdown times $\{\xi_{(1)},\ldots,\xi_{(k)}\}$ and $\{\xi'_{(1)},\ldots,\xi'_{(k)}\}$ corresponding to the areas N and $N_1 = \rho N$. Further, we estimate b_N, a_N and b_{N_1} using (3.3.13)-(3.3.14). Then the estimators $\hat{\lambda}$, $\hat{b}$ and $\hat{N}$ are found from the equations

$$(5.3.30)\begin{cases} \hat{b}_N = \frac{1}{b}\exp\{\frac{a}{2} - 1 - x^{1/2}\} \\ \frac{\hat{b}_{N_1}}{\hat{b}_N} = \exp\{x^{1/2} - (x + 2a\ln\rho)^{1/2}\} \\ \frac{\hat{a}_N}{\hat{b}_N} = \frac{1}{a}(x^{1/2}+1)\exp\{x^{-1/2}[\frac{a\ln x}{4} - \frac{1}{2} - a\phi(x^{1/2}+1 - \frac{a}{4}x^{-1/2}\ln x)]\} \end{cases}$$

where a, x and $\phi(q)$ are given by (5.2.85)-(5.2.86). This system can be solved by expressing $x^{1/2}$ via a using the second equation.

The histogram of the modified interbreakdown times $[(\xi_{(i+1)}-\xi_{(i)})\cdot i]$, corresponding to this model must be of the exponential form with mean and standard deviation equal to $\alpha\exp\{-(\beta+\ln S)^{1/2}\}$, where S is the area of the insulator and α, β are appropriate constants.

Finally, let us note, that for large m the observations $\xi_{(m+1)},\xi_{(m+2)},\ldots,\xi_{(m+k)}$ behave as they would be successive arrival times in the homogenious Poisson process (See Theorem 3.2.1).

CHAPTER VI

6. A MORE GENERAL MODEL OF BREAKDOWN. A TAUBERIAN THEOREM

In this chapter we consider the cases when a charge accumulation process in a given cell is more complicated than those considered before. In section 6.1 we consider the infinite charge breakdown model when the charge accumulation process is a general transient birth-death process.

In section 6.2 we consider a Tauberian theorem which enables one to find a representation for $F(t)$ (as $t \downarrow 0$) in terms of its Laplace transform. Such a representation may be especially useful in the cases that F, the d.f. of the time to explosion of a single cell, corresponds to an infinite convolution of independent random variables. In section 6.3 we consider an important special case of the Tauberian theorem.

6.1 The infinite charge model based on transient birth-death process

In the present model we suppose that there are N cells operating independently and the initial charge in every cell is zero. The charge accumulation in every cell proceeds in accordance with a birth-death process, λ_i $(i = 0,1,\ldots)$ being the birth rates and μ_i $(i = 1,2,\ldots)$ the death rates. We suppose that these rates are strictly positive. A cell explodes as soon as the charge accumulated in it reaches infinity. Conclusions about both regenerative and non-regenerative cases of this model may be drawn from the following considerations.

Suppose that the charge accumulation birth-death process is transient, so that the potentials $\pi_o, \pi_1, \ldots$ (See (D.8) of Appendix D) satisfy

$$(6.1.1) \qquad \sum_{i=o}^{\infty} \pi_i = \infty \quad ; \quad \sum_{i=o}^{\infty} \frac{1}{\lambda_i \pi_i} < \infty \quad .$$

Moreover, we assume that

$$(6.1.2) \qquad \sum_{i=o}^{\infty} \pi_i \sum_{j=i}^{\infty} \frac{1}{\lambda_j \pi_j} < \infty \quad ,$$

so that by Theorem D5 of Appendix D, the underlying process is transient of type 1 and

$$(6.1.3) \qquad \sum_{j=o}^{\infty} P_{ij}(t) < 1$$

holds for every state i and every $t > 0$. The last inequality implies (See Theorem D 4) that whatever the initial state of the cell, the breakdown of this cell occurs in finite time with probability one. Since $\pi_i \sum_{j=1}^{\infty} \frac{1}{\lambda_j \pi_j}$ is the mean time spent by the cell in the state i before it explodes (initial state of the cell is 0), (6.1.2) is just a generalization of the Feller-Lundberg condition (See 5.1 of Chapter V) relevant in pure birth models.

Let us denote by ξ_i the time to explosion in the $i^{\underline{th}}$ cell. Denote the distribution function of ξ_i by F(t), the corresponding density function by f(t) and the Laplace-Stieltjes transform of F(t) by $\tilde{f}(s)$. To solve the domain

of attraction problem we must find the asymptotic form of $F(t)$ as $t \downarrow 0$. We shall show that (analogously to the case of the pure birth infinite model) ξ_i is represented by infinite convolution of independent exponential random variables. This will imply that $F(t)$ belongs to the domain of attraction of L and the normalizing constants can be found using Theorem 5.1.3.

To begin with let us define the class of Polya frequency functions, which play an important role in applications (See Keilson (1979)).

Definition 6.1.1 A density function $f(x)$ is a Polya frequency function of infinite order (write $f \in PF_\infty$) if, possibly after a linear transformation, $f(x)$ is the limit of a sequence of densities each of which is a convolution of a finite number of exponential densities.

Alternative definitions, based on the concept of total positivity may be found in Karlin (1968, Chapter 6). The class PF_∞ contains many known density functions, for example the uniform and Erlang's density functions. The PF_∞ density functions posses many interesting properties. For example, it can be shown that if $f \in PF_\infty$ then $f(x)$ must be log-concave, strongly unimodal (i.e. f is unimodal and its convolution with any unimodal density is again unimodal) and has at most an exponentially decreasing tail as $x \to \infty$.

In terms of Laplace transforms $f \in PF_\infty$ means that the reciprocal of its Laplace transform $\psi(s) = 1/\tilde{f}(s)$ can be represented as limit of sequence of polynomials possesing real and negative zeros only. But such functions ψ may take only a specified form as the following theorem shows (See Karlin (1968, p. 336) for a proof).

Theorem 6.1.1 Let $\psi(x)$, for which $\psi(0)= 1$, denote a function defined for $-A \leq x \leq A$ $(A > 0)$ obtained as a limit of polynomials $P_n(x)$ possessing only real negative zeros,

$$(6.1.4) \qquad P_n(x) = \prod_{i=1}^{n} (1 + \frac{1}{\lambda_{n,i}} x) \quad , \quad \lambda_{n,i} > 0.$$

Then $\psi(x)$ admits the analytic continuation $\psi(s)$ in the complex plane and

$$(6.1.5) \qquad \psi(s) = e^{\delta s} \sum_{i=1}^{\infty} (1 + \frac{1}{\lambda_i} s) \quad , \delta \geq 0 \quad , \lambda_i > 0.$$

$$(6.1.6) \qquad \sum_{i=1}^{n} \frac{1}{\lambda_{n,i}} \xrightarrow[n\to\infty]{} \sum_{i=1}^{\infty} \frac{1}{\lambda_i} + \delta$$

Conversely, any function $\psi(s)$ of the form (6.1.5) is uniformly approximated in any bounded domain by polynomials possessing only real negative zeros.

□

Remark 1. It follows from the familiar Hurwitz theorem concerning convergent sequences of analytic functions (See Titchmarsh, 1964, p. 119) that every λ_i in (6.1.5) is a cluster point of the set $\{\lambda_{n,i} | n = 1,2,\ldots, \quad i = 1,2,\ldots,n\}$.

Remark 2. We know that if a density function f is just an infinite convolution of exponential densities, that is

$$(6.1.7) \qquad P_n(x) = \prod_{i=1}^{n} \left(1 + \frac{1}{\lambda_i} x\right)$$

then the corresponding limiting function has the form (6.1.5) with $\delta = 0$. But, if the rates λ_i in (6.1.7) depend on n the multiplier $e^{\delta s}$ may appear in the representation of $\psi(s)$. This means that the corresponding density function is an infinite convolution of exponential densities shifted by δ i.e. δ is its left endpoint.

Now let us turn our attention to the random variable ξ which represents the first passage time from the zero state to the absorbing infinity state. Denoting by T_i the first passage time from the state i to i+1 we obtain

$$(6.1.8) \qquad \xi = T_o + T_1 + \ldots + T_n + \ldots$$

Of course, the random variables $T_o, T_1, \ldots$ are independent. As we know, (See section 4.1.1) their Laplace transforms $\tilde{f}_o(s), \tilde{f}_1(s), \ldots$ satisfy the recursive relations

$$(6.1.9) \qquad \tilde{f}_n(s) = \frac{\lambda_n}{s+\lambda_n+\mu_n(1-\tilde{f}_{n-1}(s))} \quad ; \qquad n = 1,2,\ldots$$

with

$$(6.1.10) \qquad \tilde{f}_o(s) = \frac{\lambda_o}{s+\lambda_o}$$

Let us denote the first passage time from 0 to n by T_{on} and the corresponding Laplace transform by $\tilde{f}_{on}(s)$. Then

$$(6.1.11) \qquad \tilde{f}_{on}(s) = \tilde{f}_o(s)\tilde{f}_1(s) \ldots \tilde{f}_{n-1}(s) = \tilde{f}_{o,n-1}(s)\tilde{f}_{n-1}(s)$$

The following theorem (See Keilson (1979, p. 59)) shows that T_{on} can be represented as a convolution of n independent exponential random variables.

<u>Theorem 6.1.2</u> $\tilde{f}_{on}(s)$ can be represented as follows

$$(6.1.12) \qquad \tilde{f}_{on}(s) = \frac{\theta_{n,1}}{\theta_{n,1}+s} \quad \frac{\theta_{n,2}}{\theta_{n,2}+s} + \ldots + \frac{\theta_{n,n}}{\theta_{n,n}+s} \quad ,$$

where $\theta_{n,i}$ $(i = 1,2,\ldots,n)$ are distinct and positive.

<u>Proof</u>: The proof can be carried out by induction after we prove auxiliary statements a) and b) : a) Except for singularities, $\tilde{f}_n(s)$ is monotonically decreasing for real s.

b) $f_n(s)$ is a rational function, and has a simple pole between each pair of neighboring poles of $\tilde{f}_{n-1}(s)$. All (n+1) poles of $\tilde{f}_n(s)$ are on the negative halfline.

The full proof may be found in the mentioned work of Keilson.

□

Now we are able to prove the following.

Theorem 6.1.3 The time to explosion of a single cell ξ can be represented as infinite convolution of independent exponential random variables.

Proof: We see from the Theorem 6.1.2 that the density function f(t) of ξ belongs to the class PF_∞ and therefore, by Theorem 6.1.1 its Laplace transform has the form

$$(6.1.13) \qquad \tilde{f}(s) = e^{-\delta s} \prod_{i=1}^{\infty} \frac{\theta_i}{\theta_i + s} \quad ; \quad \theta_i > 0 \quad ; \quad \delta \geq 0.$$

The last equality means that f(t) is an infinite convolution of exponential densities, shifted by δ , that is

$$(6.1.14) \qquad P\{\xi < \delta\} = 0$$

But, by statement D.1 (See Appendix D) (6.1.14) implies that $\delta = 0$. So, the theorem is proved.

□

Remark From the recurrence relations (6.1.9)-(6.1.10) we can easily obtain, that

$$(6.1.15) \qquad ET_i = \frac{1}{\lambda_i \pi_i} \sum_{j=o}^{i} \pi_j \quad ,$$

so that the expected time to explosion is given by

$$(6.1.16) \qquad E\xi = \sum_{i=o}^{\infty} \frac{1}{\lambda_i \pi_i} \sum_{j=o}^{i} \pi_j = \sum_{i=o}^{\infty} \frac{1}{\theta_i}$$

An interesting fact is that ξ may be represented by another infinite sum of exponential random variables which are not independent:

$$(6.1.17) \qquad \xi = T_o' + T_1' + \ldots \quad ,$$

where T_j' is the total time spent by the process in the state i before it drifts to infinity. (The initial state of the process is supposed to be 0). The exponentiality of T_j' follows from the fact that once the process leaves the state j it can return to it with probability $0 < p < 1$ or not return with probability (1-p). The mean of T_j' is given by

$$(6.1.18) \quad ET_j' = \int_o^{\infty} P_{oj}(t)dt = \int_o^{\infty} \{\pi_j \int_o^{\infty} e^{-xt} Q_o(x) Q_j(x) d\phi(x)\}dt$$

$$= \pi_j \int_o^{\infty} \frac{Q_j(x)}{x} \, d\phi(x) = \pi_j \sum_{i=j}^{\infty} \frac{1}{\lambda_i \pi_i} \quad ,$$

(See Karlin and McGregor (1957b, p. 397)). Consequently,

$$(6.1.19) \qquad E\xi = \sum_{j=o}^{\infty} \pi_j \sum_{i=j}^{\infty} \frac{1}{\lambda_i \pi_i} \quad ,$$

and this is equivalent to (6.1.16) if the order of summation is interchanged.

Now the domain of attraction problem for F(t) can be solved exactly as in the general case of pure birth process (See 5.3). First of all, we see that F(t) belongs to the domain of attraction of L and

$$(6.1.20) \qquad F(\lambda(r)) \sim \frac{1}{\sqrt{2\pi}} \frac{\Lambda(r)}{r} \quad , \quad \text{as} \quad r \to \infty \quad (\lambda(r) \to 0).$$

where

$$(6.1.21) \qquad \Lambda(r) = \frac{e^{r\lambda(r)}\tilde{f}(r)}{\sigma(r)} \quad ;$$

$$(6.1.22) \qquad \lambda(r) = \sum_{j=o}^{\infty} (\theta_j + r)^{-1} \quad ; \quad \sigma(r) = \left(\sum_{j=o}^{\infty} (\theta_j + r)^{-2}\right)^{1/2}.$$

The results are sumed up in the following

<u>Theorem 6.1.4</u> F(t) belongs to the domain of attraction of L(t), the sequence of normalizing constants $\{a_N\}$ and $\{b_N\}$ being determined by

$$(6.1.23) \qquad \begin{cases} b_N = \dfrac{1}{r_N} \\ a_N = \lambda(r_N) \end{cases} ,$$

where r_N is the root of equation

$$(6.1.24) \qquad \frac{1}{\sqrt{2\pi}} \frac{\Lambda(r)}{r} = \frac{1}{N} \quad .$$

□

The last theorem does not seem very useful for determining the normalizing constants since the computation of θ_i requires considerable effort (for example, θ_1 is the limit of sequence $\{\theta_{n,i}\}$ defined by Theorem 6.1.3). But this is not so - we need not compute the θ_i's to use Theorem 6.1.4. Since

$$(6.1.25) \qquad \lambda(r) = -[\ln \tilde{f}(r)]' ,$$

$$(6.1.26) \qquad \sigma(r) = (-\lambda'(r))^{1/2} = \{[\ln f(r)]''\}^{1/2}$$

and, by (6.1.8) - (6.1.10)

$$(6.1.27) \qquad \tilde{f}(r) = \prod_{i=o}^{\infty} \tilde{f}_i(r) ,$$

the normalizing sequences may be found to any degree of accuracy directly from the birth-death rates.

6.2 A Tauberian theorem

The method used by Hirshchman and Widder to estimate the left tail of f(t) and its derivatives (f(t) is a density function of the infinite convolution of exponential random variables) is based on the fact that the function

$$(6.2.1) \quad E_r(s) = \prod_{j=o}^{\infty} [1 - \frac{s}{\sigma(r)(\lambda_j+r)}] \, e^{\frac{s}{\sigma(r)(\lambda_j+r)}} \quad (\sigma^2(r) = \sum_{j=o}^{\infty} (\lambda_j+r)^{-2})$$

tends to $e^{-s^2/2}$ as $r \to \infty$ (See Sec. 5.1.3 of Chapter V).

In this paragraph we show how the idea of their proof may be used to obtain a Tauberian theorem, expressing the left tail of a general distribution function $F(t)$ in the terms of its Laplace transform. In the next paragraph we shall apply this theorem to the case that $F(t)$ is a d.f. of an infinite convolution of non-exponential densities.

Let $F(t)$ be a d.f. of a random variable T and $\tilde{F}(s)$ be its Laplace transform. Suppose that the left endpoint of $F(t)$ is zero (all the conclusions may be easily re-formulated for the case that the left endpoint is a non-zero number or $(-\infty)$).

To express the values of $F(t)$ as $t \downarrow 0$ via the Laplace transform $\tilde{F}(s)$ we proceed as follows:

a) We "stretch" $F(t)$ along the t-axis by considering the function $F(t\eta(r))$ instead of $F(t)$ ($\eta(r)$ is a continuous function and $\eta(r) \to 0$ as $r \to \infty$).

b) The part of $F(t)$ not connected with its left tail is "killed" by considering the function $F(t\eta(r))$ multiplied by $\exp\{-t\psi(r)\}$, where ψ takes non-negative values only. The function $c(r)$ is chosen so that

$$(6.2.2) \qquad g_r(t) = c(r) \cdot \frac{1}{\tilde{F}(r)} \cdot F(t\eta(r))e^{-t\psi(r)} \quad , \quad t > 0$$

will be a proper density function of some positive random variable T_r for arbitrary $r > 0$. The function ψ is chosen so that

$$(6.2.3) \qquad ET_r = o\left[\frac{1}{\eta(r)}\right] \quad \text{as} \quad r \to \infty$$

(6.2.4) $$\text{Var } T_r = O(1) \quad \text{as } r \to \infty.$$

Since $\tilde{g}_r(s)$, the Laplace transform of $g_r(t)$, is given by

(6.2.5) $$\tilde{g}_r(s) = \{\tilde{F}[\frac{\psi(r)}{\eta(r)}]\}^{-1}\, \tilde{F}[\frac{\psi(r)}{\eta(r)} + \frac{s}{\eta(r)}] \quad ,$$

by choosing $\psi(r)$ equal to $r\eta(r)$ we obtain

(6.2.6) $$\tilde{g}_r(s) = \frac{\tilde{F}[r + \frac{s}{\eta(r)}]}{\tilde{F}(r)} \quad ,$$

(6.2.7) $$ET_r = -\frac{1}{\eta(r)} \cdot \frac{\tilde{F}'(r)}{\tilde{F}(r)}$$

(6.2.8) $$\text{Var } T_r = \frac{1}{[\eta(r)]^2} \cdot \frac{\tilde{F}(r)\tilde{F}''(r) - [F'(r)]^2}{[\tilde{F}(r)]^2}$$

(6.2.9) $$c(r) = \eta(r)$$

c) If it is known that $\lim_{r\to\infty} g_r[ET_r]$ exists, the values of $F(t)$ for small arguments may be obtained from (6.2.2) after fixing t at the point $t = ET_r$.

Since for any positive functions f and g the relation $f(t)/g(t) \xrightarrow[t\to 0]{} 0$ implies that $\tilde{f}(s)/\tilde{g}(s) \to 0$ as $s \to \infty$(See Doetsch(1974, p. 221)), (6.2.3) follows from the relation (6.2.7).

After setting

$$(6.2.10) \qquad \xi(r) = -\frac{\tilde{F}'(r)}{\tilde{F}(r)} = -[\ln \tilde{F}(r)]' \quad ,$$

we define $\eta(r)$ as follows

$$(6.2.11) \quad \eta(r) = \{ \frac{\tilde{F}(r)\tilde{F}''(r)-[F'(r)]^2}{[\tilde{F}(r)]^2} \}^{1/2} = \{- \xi'(r)\}^{1/2} \quad .$$

So, denoting the centered version of the density function $g_r(t)$ and its Laplace transform by $\overline{g}_r(t)$ and $\tilde{\tilde{g}}_r(s)$ respectively; i.e.

$$(6.2.12) \qquad \overline{g}_r(t) = g_r(t + \frac{\xi(r)}{\eta(r)}) \quad ,$$

the results may be presented as follows:

<u>Lemma 6.2.1</u> Let $F(t)$ be a distribution function with 0 being its left endpoint and let $\tilde{F}(s)$ be its Laplace transform. If $\xi(r)$ and $\eta(r)$ are defined by (6.2.10)-(6.2.11) then for every $r > 0$

$$(6.2.13) \qquad \tilde{\tilde{g}}_r(s) = \frac{\tilde{F}[r + \frac{s}{\eta(r)}]}{\tilde{F}(r)} \cdot e^{\frac{s\xi(r)}{\eta(r)}}$$

is a (bilateral) Laplace transform of the random variable $\overline{T}_r$ such that

$E\overline{T}_r = 0$, $\text{Var}\ \overline{T}_r = 1$ and the density function $\overline{g}_r(t)$ of $\overline{T}_r$ is given by

$$(6.2.14) \qquad \overline{g}_r(t) = \frac{\eta(r)e^{-r\xi(r)}}{\tilde{F}(r)} F(t\eta(r) + \xi(r))e^{-tr\eta(r)}$$

□

After putting t=0 in (6.2.14) we obtain

$$(6.2.15) \qquad F(\xi(r)) = \overline{g}_r(o) \frac{e^{r\xi(r)} \cdot \tilde{F}(r)}{\eta(r)}$$

It might be expected that under some general conditions the sequence $\overline{g}_r(o)$ will tend as $r \to \infty$ to a positive limit $\overline{g}_o$. Then (6.2.15) leads to an asymptotic expression for $F(\xi(r))$ in terms of $\tilde{F}(r)$. For example, if $F(t)$ is the d.f. of an infinite convolution of exponential terms with rates $\{\lambda_j,\ j = 0,1,\ldots\}$ the sequence $\{\tilde{\overline{g}}_r(s)\}$ may be shown to converge to $e^{s^2/2}$ uniformly in any bounded region, this implies that $\overline{g}_o = 1/\sqrt{2\pi}$ and we obtain that

$$(6.2.15a) \qquad F(\xi(r)) \sim \frac{1}{\sqrt{2\pi}} \frac{e^{r\xi(r)} \cdot \tilde{F}(r)}{\eta(r)} \quad , \text{ as } r \to \infty$$

(See Sec. 5.1.3 of Chapter V.). This form is slightly different from that given by Lemma 5.1.3 because $\xi(r)$ and $\eta(r)$ differ from the functions $\lambda(r)$ and $\sigma(r)$ respectively:

$$(6.2.16) \qquad \xi(r) = \frac{1}{r} + \sum_{j=o}^{\infty} \frac{1}{\lambda_j + r} = \lambda(r) + \frac{1}{r}$$

$$(6.2.17) \qquad [\eta(r)]^2 = -\xi'(r) = \frac{1}{r^2} + [\sigma(r)]^2 \quad .$$

Since

$$\bar{g}_r(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\tilde{g}}_r(-is)ds,$$

the results can be formulated by means of the following

Theorem 6.2.1 If

$$\int_{-\infty}^{\infty} \tilde{\tilde{g}}_r(-is)ds \longrightarrow \bar{g}(o) > 0, \quad \text{as} \quad r \to \infty \quad .$$

then, as $r \to \infty$

$$(6.2.18) \qquad F(\xi(r)) \sim \bar{g}(o) \frac{e^{r\xi(r)}\tilde{F}(r)}{\eta(r)} \quad ,$$

where $\xi(r)$ and $\xi(r)$ are defined by (6.2.10)-(6.2.11)

□

The following two simple examples illustrate the application of this theorem.

Example 1 Let us consider the Laplace transform

$$(6.2.19) \qquad \tilde{F}(s) = \frac{\lambda}{s(\lambda+s)} \quad ,$$

corresponding to a d.f. $F(t) = 1 - e^{-\lambda t}$ $(t \geq 0)$.

In this case

$$(6.2.20) \qquad \xi(r) = \frac{1}{r} + \frac{1}{\lambda+r} \quad ;$$

$$(6.2.21) \qquad \eta(r) = \frac{1}{r}[1 + (\frac{r}{\lambda+r})^2]^{1/2}$$

$$(6.2.22) \qquad \tilde{\tilde{g}}_r(s) = \{1 + \frac{s}{[1+(\frac{r}{\lambda+r})^2]^{1/2}}\}^{-1} \cdot \{1 + \frac{s}{[1+(\frac{\lambda+r}{r})^2]^{1/2}}\}^{-1} \times$$

$$\times \exp\{s\frac{1 + \frac{r}{\lambda+r}}{[1+(\frac{r}{\lambda+r})^2]^{1/2}}\} \to \tilde{\tilde{g}}(s) = (1 + \frac{s}{\sqrt{2}})^{-2} e^{s\sqrt{2}} \quad \text{as } r \to \infty .$$

The density function $\overline{g}(t)$, corresponding to $\tilde{\tilde{g}}(s)$ is

$$\overline{g}(t) = 2(t+\sqrt{2})e^{-\sqrt{2}(t+\sqrt{2})}, \quad t \geq -\sqrt{2}$$

It can be shown that the conditions of Theorem 6.2.1 are satisfied with $\overline{g}(o) = 2\sqrt{2}\, e^{-2}$; then (6.2.18) gives

$$(6.2.23) \quad F(\frac{1}{r} + \frac{1}{\lambda+r}) \sim 2\sqrt{2}\, e^{-2} \frac{e^{1 + \frac{r}{r+\lambda}} \cdot \lambda}{[(\lambda+r)^2+r^2]^{1/2}} \sim \frac{2\lambda}{r} \quad \text{as} \quad r \to \infty$$

and after denoting $\xi(r)$ by t we finally obtain

$$F(t) \sim \lambda t \quad \text{as} \quad t \downarrow 0 \quad ,$$

the true asymptotic representation of F(t).

<u>Example 2</u> Consider the Laplace transform

$$\tilde{F}(s) = \frac{1}{s} e^{s^2/2} \quad ,$$

corresponding to a standard normal distribution function $\Phi(t)$.

In this case $\xi(r)$ must tend to $(-\infty)$ as $r \to \infty$ (See 6.2.2) and we obtain

$$\xi(r) = \frac{1}{r} - r \quad ;$$

$$\eta(r) = \left(\frac{1}{r^2} + 1\right)^{1/2} \quad ;$$

$$(6.2.24) \qquad \tilde{\tilde{g}}_r(s) = \frac{r\eta(r)}{r\eta(r)+s} \cdot e^{\frac{1}{2}\{(r + \frac{s}{\eta(r)})^2 - r^2\}} \cdot e^{s\,\frac{\frac{1}{r} - r}{\eta(r)}} \to$$

$$\to \tilde{\tilde{g}}(s) = e^{s^2/2} \quad \text{as} \quad r \to \infty$$

The density function $\overline{g}(t)$, corresponding to $\tilde{\tilde{g}}(s)$ is a standard normal one

and the conditions of Theorem 6.2.1 are satisfied with $\overline{g}(o) = 1/\sqrt{2\pi}$. Thus, (6.2.18) gives

$$(6.2.25) \quad \Phi(\xi(r)) \sim \frac{1}{\sqrt{2\pi}} \frac{e^{1-r^2} \cdot (1/r) e^{r^2/2}}{(\frac{1}{r^2} + 1)^{1/2}} \sim \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{1-r^2/2}}{r} \quad \text{as} \quad r \to \infty \quad ,$$

and after denoting $\xi(r)$ by t we obtain

$$\Phi(t) \sim \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(-t)} e^{-t^2/2} \quad \text{as} \quad t \to -\infty ,$$

the true asymptotic expression (See Dingle (1973)).

To deal with the domain of attraction problem denote

$$(6.2.26) \qquad F^{(-n)}(t) = \int_o^t F^{(-n+1)}(t)dt \quad ; \quad F^{(o)}(t) = F(t) \ .$$

Then the Laplace transform of $F^{(-n)}(t)$ is $\tilde{F}(s)/s^n$. Since the growth of $F^{(-n)}(t)$ as $t \to \infty$ is controlled by that of $F(t)$, Lemma 6.2.1 can be easily modified for $F^{(-n)}(t)$ instead of $F(t)$ and we obtain that the Laplace transform of the density function

$$(6.2.27) \qquad \overline{g}_{r,n}(t) = \frac{r^n \eta(r) e^{-r\xi(r)}}{\tilde{F}(r)} F^{(-n)}(t\eta(r)+\xi(r)) e^{-tr\eta(r)}$$

is given by

$$(6.2.28)\quad \tilde{\tilde{g}}_{r,n}(s) = \frac{\tilde{F}[r + \frac{s}{\eta(r)}]}{\tilde{F}(r)} \cdot e^{\frac{s\xi(r)}{\eta(r)}} \cdot \left[\frac{1}{1 + \frac{s}{r\eta(r)}}\right]^n .$$

Now we can prove the following

<u>Theorem 6.2.2</u> Suppose that $r\eta(r) \to \infty$ as $r \to \infty$. As well, suppose that there exists a function $\tilde{\tilde{g}}(s)$ which is a (bilateral) Laplace transform of some $\bar{g}(t)$ and satisfies the conditions

$$(6.2.29)\qquad \text{(i)} \int_{-\infty}^{\infty} \tilde{\tilde{g}}(-is)ds > 0$$

and

$$(6.2.30)\qquad \text{(ii)} \int_{-\infty}^{\infty} (\tilde{\tilde{g}}_r(-is)-\tilde{\tilde{g}}(-is))e^{-ist}ds \to 0 \quad \text{as} \quad r \to \infty \quad ,$$

uniformly for $t \in [-a,a]$ (a is an arbitrary small positive number).

Then $F(t)$ belongs to the domain of attraction of L with normalizing constants a_N (shift) and b_N (scale) given by

$$(6.2.31)\qquad \begin{cases} b_N = \dfrac{1}{r_N} \\ a_N = \xi(r_N) \end{cases} ,$$

where r_N is the solution of

$$(6.2.32) \qquad \overline{g}(o)\,\frac{e^{r\xi(r)}\tilde{F}(r)}{\eta(r)} = \frac{1}{N}\,.$$

<u>Proof</u>: The densities $\overline{g}_r(t)$ converge to $\overline{g}(t)$ uniformly in $[-a,a]$, as can be seen from (6.2.30). Further, (6.2.28) shows that $\tilde{\tilde{g}}_{r,n}(s)$ is a Laplace transform of $\overline{T}_r$, shifted n times by an exponential random variable with parameter $r\eta(r)$.

Consider the case n=1. For $t \in [-\frac{a}{2}, \frac{a}{2}]$ and every $\alpha > 0$ we have that

$$(6.2.33) \qquad \overline{g}_{r,1}(t) = \int_{-\infty}^{t} \overline{g}_r(z) \cdot r\eta(r)e^{-r\eta(r)(t-z)}dz =$$

$$= \int_{-\infty}^{t-\alpha} \overline{g}_r(z) \cdot r\eta(r)e^{-r\eta(r)(t-z)}dz +$$

$$+ \int_{t-\alpha}^{t} [\overline{g}_r(z) - \overline{g}(z)] \cdot r\eta(r)e^{-r\eta(r)(t-z)}dz +$$

$$+ \int_{t-\alpha}^{t} [\overline{g}(z) - \overline{g}(t)] \cdot r\eta(r)e^{-r\eta(r)(t-z)}dz + \overline{g}(t)\int_{t-\alpha}^{t} r\eta(r)e^{-r\eta(r)(t-z)}dz,$$

Since $\overline{g}(t)$ is continuous in $[-\frac{a}{2}, \frac{a}{2}]$ (as a limit function of a uniformly convergent sequence of continuous functions) α can be chosen so that for given $\varepsilon > 0$ and every $t \in [-\frac{a}{2}, \frac{a}{2}]$

$$|\overline{g}(z) - \overline{g}(t)| < \frac{\varepsilon}{3} \qquad \text{as} \quad t - \alpha < z < t$$

Now we see that if α is chosen in such a way then for every $t \in [-\frac{a}{2}, \frac{a}{2}]$

$$(6.2.34) \quad |\overline{g}_{r,1}(t)-\overline{g}(t)| \leq e^{-r\eta(r)\alpha} + \int_{t-\alpha}^{t} |\overline{g}_r(z)-\overline{g}(z)| r\eta(r) e^{-r\eta(r)(t-z)} dz +$$

$$+ \frac{\varepsilon}{3}[1 - e^{-r\eta(r)\alpha}] + \overline{g}(t)e^{-r\eta(r)\alpha} .$$

Since there exist R^* such that for every $t \in [-\frac{a}{2}, \frac{a}{2}]$

$$(6.2.35) \qquad |\overline{g}_r(t) - \overline{g}(t)| < \frac{\varepsilon}{3}$$

and

$$(6.2.36) \qquad [1 + \overline{g}(t)]e^{-r\eta(r)\alpha} < \frac{\varepsilon}{3}$$

whenever $r > R^*$, $\overline{g}_{r,1}(t)$ tends to $\overline{g}(t)$ as $r \to \infty$, uniformly in the interval $t \in [-\frac{a}{2}, \frac{a}{2}]$. Therefore $\overline{g}_{r,1}(o) = \overline{g}(o)$ and (6.2.7) gives that as $r \to \infty$,

$$(6.2.37) \qquad F^{(-1)}(\xi(r)) \sim \overline{g}(o) \frac{e^{r\xi(r)}\tilde{F}(r)}{\eta(r)} \cdot \frac{1}{r}$$

Further, we can show by induction that an asymptotic equivalence of the type (6.2.37) holds for every integer n:

$$(6.2.38) \qquad F^{(-n)}(\xi(r)) \sim \overline{g}(o)\ \frac{e^{r\xi(r)}\tilde{F}(r)}{\eta(r)} \cdot \frac{1}{r^n} \quad , \qquad r \to \infty .$$

Since

$$(6.2.39) \qquad \frac{F(\xi(r))F^{(-2)}(\xi(r))}{[F^{(-1)}(\xi(r))]^2} \longrightarrow 1 \quad \text{as} \quad r \to \infty \quad ,$$

by De Haan's theorem (See Theorem 3.1.9) F belongs to the domain of attraction of L. The expressions for normalizing constants can be obtained exactly as in Theorem 5.1.3.

□

<u>Remark</u> The condition (6.2.30) is satisfied if

$$(6.2.40) \qquad \int_{-\infty}^{\infty} |\tilde{\tilde{g}}_r(-is) - \tilde{\tilde{g}}(-is)|ds \longrightarrow 0 \quad \text{as } r \to \infty \quad .$$

6.3 Special case

In this paragraph we show that if $\tilde{F}(s)$ satisfies some general conditions, then the sequence $\{\tilde{\tilde{g}}_r(s)\}$ converge to $\exp\{s^2/2\}$, the Laplace transform of the standard normal variate, and the Tauberian relation between $\tilde{F}(s)$ as $r \to \infty$ and $F(t)$ as $t \downarrow 0$ is given by (6.2.15a). We can write (by 6.2.13)

$$(6.3.1) \qquad \ln \tilde{\tilde{g}}_r(s) = \ln \tilde{F}(r + \frac{s}{\eta(r)}) - \ln \tilde{F}(r) + \frac{s\xi(r)}{\eta(r)} \quad ,$$

and, expanding the first term, we obtain (See (6.2.10) and (6.2.11))

$$\ln \tilde{\tilde{g}}_r(s) = \frac{s^2}{2} - \frac{1}{3!}\,\xi''\left(r + \frac{\theta s}{\eta(r)}\right) \cdot \frac{s^3}{\eta^3(r)} \quad , \tag{6.3.2}$$

where $0 \leq \theta \leq 1$. The last expression leads to a proof of the following

<u>Lemma 6.3.1</u> If

$$\frac{\xi''\left(r + \frac{s^*}{\eta(r)}\right)}{\eta^3(r)} \longrightarrow 0 \quad \text{as } r \to \infty \quad , \tag{6.3.6}$$

uniformly for s^* in any bounded region then

$$\tilde{\tilde{g}}_r(s) \to e^{s^2/2} \quad \text{as} \quad r \to \infty \quad , \tag{6.3.7}$$

uniformly for s being in any bounded region.

□

Now we shall prove a special case of the Theorem 6.2.2.

<u>Theorem 6.3.1</u> If $\tilde{F}$ satisfies (6.3.6), $r\eta(r) \to \infty$ as $r \to \infty$ and $\int_{-\infty}^{\infty} |\tilde{\tilde{g}}_r(-is) - e^{-s^2/2}|\,ds \to 0$ as $r \to \infty$, then F belongs to the domain of attraction of L,

$$F(\xi(r)) \sim \frac{1}{\sqrt{2\pi}}\,\frac{e^{r\xi(r)}\tilde{F}(r)}{\eta(r)} \quad \text{as} \quad r \to \infty \quad , \tag{6.3.8}$$

and the normalizing constants are given by

$$(6.3.9) \qquad \begin{cases} b_N = \dfrac{1}{r_N} \\ \\ a_N = \xi(r_N) \end{cases} ,$$

where r_N is the solution of

$$(6.3.10) \qquad \frac{1}{\sqrt{2\pi}} \frac{e^{r\xi(r)}\tilde{F}(r)}{\eta(r)} = \frac{1}{N} .$$

<u>Proof</u>: The proof is similar to that of Theorem 6.2.2 and is therefore omitted.

□

The last theorem may be especially useful when F is the d.f. of an infinite convolution, since for this case the functions $\xi(r)$ and $\eta^2(r)$ may be represented as sums of functions, each depending on the distribution of the particular term of the convolution only. So, $\xi(r)$ and $\eta^2(r)$ may be evaluated (as $r \to \infty$) using one of the summation methods.

CHAPTER VII

7. CONCLUSIONS

In this chapter we give a resume and formulate several unsolved problems which are of interest in connection with the present work.

7.1 Resume

In this section we sketch briefly the results of our research.

In Chapter I we supply information concerning the physical aspects of breakdown phenomena and discuss the ways it has been treated in the literature.

In Chapter II we present a class of stochastic models which is the main object of investigation in the thesis. All of them are "weakest link" models with a birth-death process used to describe charge accumulation in a single cell. The models were classified as regenerative or non-regenerative on the one hand and as finite or infinite charge breakdown models on the other. We showed that there is experimental evidence supporting the relevance of these models.

In Chapter III relevant results from Extreme Value Theory were provided and general methods of identification of the appropriate breakdown mechanism and estimation of its parameters were discussed.

Chapter IV was devoted to the study of finite charge breakdown models. It is shown that in this case F(t), the distribution function of the time

to explosion of a single cell, belongs to the domain of attraction of Weibull's distribution law and normalizing constants are found. Both regenerative and non-regenerative types of the models are considered.

In Chapter V we consider the infinite charge breakdown models when the charge accumulation process in a single cell is a pure birth process. We show that in this case F(t) belongs to the domain of attraction of Gumbel's law and give equations which enable one to find the appropriate normalizing sequences. Since these equations do not provide closed expressions for the normalizing constants, we develop an alternative approach. We use it to obtain an asymptotic representation of the left tail of F(t) and to find explicit expressions for the normalizing constants in two important cases - when the birth rates are of the type $\lambda_j = a(j+1)^{\alpha}$ and $\lambda_j = b\lambda^j$. Then we consider the estimation problem for both regenerative and non-regenerative variants of these models.

Finally, in Chapter VI we show that the infinite charge breakdown model when the charge accumulation process in a single cell is a birth-death process with rates $\{\lambda_i\}$ and $\{\mu_i\}$ is equivalent to one with a pure birth charge accumulation process (the "birth" rates of the latter depend on $\{\lambda_i\}$ and $\{\mu_i\}$). We give a procedure which enables one to calculate the normalizing constants.

In this chapter we also prove a Tauberian theorem which enables one to obtain the asymptotic representation of some monotone function F(t) in terms of its Laplace transform. For the case that F(t) is a distribution function we show how its domain of attraction may be identified and the normalizing

constants may be found in terms of the Laplace transform. This theorem may be useful in cases for which the charge accumulation process in a single cell is of more complicated form.

7.2 Open problems

In this section we mention several open problems which may serve as directions of further investigation.

Problem 1 Continuous charge breakdown model

In this model we assume that the charge, accumulated in any of N cells (operating independently) at time t is a continuous random variable X(t). The breakdown in a cell occurs as soon as the accumulated charge reaches some prescribed level M (the possibility of constant delay D can easily be incorporated in the final results). The initial state of the cell is characterized by the density function $p(x)$ $(0 \leq x \leq M)$, where $p(x)dx$ is a proportion of cells with accumulated charge between x and x+dx. Suppose that the time axis is divided into intervals of length Δt and the following relation holds

$$X(t+\Delta t) - X(t) = \psi(X(t),\theta_1,\ldots,\theta_r,\varepsilon) \cdot \Delta t, \tag{7.2.1}$$

where $\theta_1,\ldots,\theta_r$ is the set of parameters, characterizing the experimental conditions to which a cell is subjected at time t when accumulated charge in it is equal to X(t), and ε is a random variable, representing the noise. When the function ψ is specified, (7.2.1) leads to stochastic differential

equation with initial condition given in terms of $p(x)$. As soon as the solution $X(t)$ of this equation is found, the distribution function of the time to explosion in a single cell is given by

$$(7.2.2) \qquad F(t) = P\{X(t) \geq M\} \quad .$$

The problem is to find an appropriate (from the physical point of view) function ψ , to solve the resulting differential equation and to analyze the extreme value properties of $F(t)$.

Problem 2 The limit function problem

Let $F_1(t), F_2(t), \ldots, F_n(t), \ldots$ be a convergent sequence of distribution functions having the same left endpoint t_o and let $F(t)$ be a limit function. Assume that all the functions in the sequence belong to the domain of attraction of $G(x)$. What are the conditions (concerning the functions of the sequence) for $F(t)$ to belong to the same domain of attraction? If $F(t)$ belongs to some domain of attraction what is the connection between its normalizing constants and those of $F_1, F_2, \ldots, F_n, \ldots$? It seems, in the light of De Haan's unifying approach (See Theorem 3.1.9), that the conditions of interest are related to the behaviour of the sequence $g_1(t), g_2(t), \ldots, g_n(t), \ldots$ in the vicinity of t_o, where $g_i(t)$ is given by (3.1.21).

Problem 3 The Tauberian relation

Suppose that $F(t)$ and $G(t)$ are distribution functions with common left endpoint $t_o=0$ and $\tilde{F}(s)$, $\tilde{G}(s)$ are their Laplace transforms. It is well known

(See Doetsch(1974), p. 221) that $\lim_{t\to 0} [F(t)/G(t)] = 1$ implies that $\lim_{s\to\infty} [\tilde{F}(s)/\tilde{G}(s)] = 1$. If it is known that $\tilde{F}(s)$, $\tilde{G}(s)$ are Laplace transforms of the continuous functions with a common left endpoint and $\tilde{F}(s) \sim \tilde{G}(s)$ as $s \to \infty$, what are the conditions that ensure $F(t) \sim G(t)$ as $t \downarrow 0$?

Problem 4

As we have seen in Section 4.1.1, in the case of finite charge breakdown models with zero initial charge in every cell, the asymptotic representation of $F(t)$ as $t \downarrow 0$ depends on the birth rates only. Under what conditions can this property be extrapolated to the case of infinite charge breakdown models?

APPENDIX A

BASIC PROPERTIES OF THETA-FUNCTIONS

1. Definition of theta-functions and basic properties table.

The four theta-functions are denoted as follows:

$$\theta_1(z|\tau) = 2 \sum_{n=o}^{\infty} (-1)^n e^{i\pi\tau(n+\frac{1}{2})^2} \sin[(2n+1)z] \tag{A.1}$$

$$\theta_2(z|\tau) = 2 \sum_{n=o}^{\infty} e^{i\pi\tau(n+\frac{1}{2})^2} \cos[(2n+1)z] \tag{A.2}$$

$$\theta_3(z|\tau) = 1 + 2 \sum_{n=1}^{\infty} e^{i\pi\tau n^2} \cos(2nz) \tag{A.3}$$

$$\theta_4(z|\tau) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{i\pi\tau n^2} \cos(2nz) \quad . \tag{A.4}$$

In the definition z is considered as variable and τ as a parameter. For the convergence of the series that are present in the definitions we demand that $\mathrm{Im}\,\tau > 0$.

We define also

$$\theta_i = \theta_i(0|\tau) \quad , \quad i = 2,3,4$$

(A.5)

$$\theta_1' = [\frac{d}{dz}\ \theta_1(z|\tau)]_{z=o} \quad .$$

As well, we shall write sometimes $\theta_i(z)$ instead of $\theta_i(z|\tau)$. $\theta_i(z)$ are "almost" double periodic functions with periods π and $\pi\tau$ as can be seen from the following basic properties table:

$y =$	z	$z + \frac{\pi}{2}$	$z + \pi$	$z + \frac{\pi\tau}{2}$	$z + \pi\tau$
$\theta_1(y)$	$\theta_1(z)$	$\theta_2(z)$	$-\theta_1(z)$	$ie^{-\frac{i\pi\tau}{4} - iz}\theta_4(z)$	$-e^{-i\pi\tau-2iz}\theta_1(z)$
$\theta_2(y)$	$\theta_2(z)$	$-\theta_1(z)$	$-\theta_2(z)$	$e^{-\frac{i\pi\tau}{4} - iz}\theta_3(z)$	$e^{-i\pi\tau-2iz}\theta_2(z)$
$\theta_3(y)$	$\theta_3(z)$	$\theta_4(z)$	$\theta_3(z)$	$e^{-\frac{i\pi\tau}{4} -iz}\theta_2(z)$	$e^{-i\pi\tau-2iz}\theta_3(z)$
$\theta_4(y)$	$\theta_4(z)$	$\theta_3(z)$	$\theta_4(z)$	$ie^{-\frac{i\pi\tau}{4} - iz}\theta_1(z)$	$-e^{-i\pi\tau-2iz}\theta_4(z)$

2. Zeros of theta-functions, the relation $\theta_1' = \theta_2\,\theta_3\,\theta_4$ and connection between $\theta_3(z|\tau)$ and $\theta_3(z/\tau|-1/\tau)$.

It can be shown that all theta-functions are entire functions and their only zeros are:

$$
\begin{aligned}
&\theta_1(z) = 0 \quad \text{at} \quad z = 0 + m\pi + n\pi\tau \\
&\theta_2(z) = 0 \quad \text{at} \quad z = \pi/2 + m\pi + n\pi\tau \\
&\theta_3(z) = 0 \quad \text{at} \quad z = \frac{\pi}{2} + \frac{\tau\pi}{2} + m\pi + n\pi\tau \\
&\theta_4(z) = 0 \quad \text{at} \quad z = \frac{\tau\pi}{2} + m\pi + n\pi\tau ,
\end{aligned}
\tag{A.6}
$$

where m and n are integers.

It is known also (See Rainville (1960)) that there exists a relation between θ_1' and θ_2, θ_3, θ_4 namely

(A.7) $$\theta_1' = \theta_2\ \theta_3\ \theta_4 .$$

The relation between $\theta_3(z|\tau)$ and $\theta_3(z/\tau|-1/\tau)$ plays a crucial role in finding the asymptotic representation of $F(t)$ at $t \downarrow 0$. Since the proof of this relation rarely appears in the literature, we shall give a short sketch of it. The full proof can be found in Whittaker and Watson (1963, p.475). The relation is:

(A.8) $$\theta_3(z|\tau) = (-i\tau)^{-1/2} \exp\left(\frac{z^2}{i\pi\tau}\right) \theta_3(z/\tau|-1/\tau).$$

To prove this equality we consider the equation

(A.9) $$\eta(z|\tau) = \frac{\theta_3(z/\tau|-1/\tau)\exp\left(\frac{z^2}{i\pi\tau}\right)}{\theta_3(z|\tau)}$$

Then, using the basic properties table we show that $\eta(z|\tau)$ is a double periodic function with periods π and $\pi\tau$:

(A.10) $$\eta(z+\pi|\tau) = \eta(z+\pi\tau|\tau) = \eta(z|\tau)$$

The last fact means that $\eta(z|\tau)$ is an elliptic function. (See Rainville (1960)). Since (by A.6) the denominator $\theta_3(z|\tau)$ has only <u>one</u> zero in the parallelogramm with the vertices. 0, π, $\pi+\pi\tau$, $\pi\tau$, the function $\eta(z|\tau)$

is an elliptic function of order <u>one</u>. But we know that elliptic function of order less than two must be a constant. Denoting this constant by C we obtain

$$\text{(A.11)} \qquad C\theta_3(z|\tau) = \exp\left(\frac{z^2}{i\pi\tau}\right)\ \theta_3(z/\tau|-1/\tau)$$

Shifting the argument z in (A.11) by $\frac{\pi}{2}$, $\frac{\pi\tau}{2}$ and $\frac{\pi}{2} + \frac{\pi\tau}{2}$, using the basic properties table and the equality $\theta_1' = \theta_2\ \theta_3\ \theta_4$, we obtain that $C = (-i\pi)^{1/2}$ and so, (A.8) is proved.

□

Now, substituting $z + \frac{\pi}{2}$ instead of z into (A.8) we obtain

$$\text{(A.12)} \qquad \theta_4(z|\tau) = (-i\tau)^{-1/2}\exp\left(\frac{z^2}{i\pi\tau}\right)\ \theta_2(z/\tau|-1/\tau).$$

APPENDIX B

EULER'S SUMMATION FORMULA

1. Bernoulli numbers

Bernoulli numbers are the coefficients $B_1, B_2 \ldots$ in the expansion

$$\frac{x}{e^x-1} = 1 - \frac{x}{2} + B_1 \frac{x^2}{2!} - B_2 \frac{x^4}{4!} + B_3 \frac{x^6}{6!} - \ldots \tag{B.1}$$

It is easy to verify that the first Bernoulli numbers are

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \tag{B.2}$$

but the higher numbers become more complicated.

2. Bernoulli polynomials

The Bernoulli polynomial of degree n, denoted by $\phi_n(x)$, is the coefficient of $t^n/n!$ in the expansion of

$$t\,\frac{e^{xt}-1}{e^t-1}$$

The first four Bernoulli polynomials are

$$\begin{aligned} \phi_1(x) &= x \\ \phi_2(x) &= x^2 - x \\ \phi_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \\ \phi_4(x) &= x^4 - 2x^3 + x^2 \end{aligned} \tag{B.3}$$

The properties of Bernoulli polynomials can be found in Bromwich ((1965), p.300).

3. Euler's summation formula

Let the function f(x) be defined in the interval [a,b], where (b-a) is any positive integer (the function f can take also complex values). Then Euler's summation formula is:

$$\text{(B.4)} \qquad f(a) + f(a+1) + \ldots + f(b) = \int_a^b f(x)dx + \frac{1}{2}\{f(a) + f(b)\} + \frac{B_1}{2!}\{f'(b) - f'(a)\} - \frac{B_2}{4!}\{f'''(b) - f'''(a)\} + \ldots + (-1)^n \frac{B_{n-1}}{(2n-2)!}\{f^{(2n-3)}(b) - f^{(2n-3)}(a)\} - R_n,$$

where

$$\text{(B.5)} \qquad R_n = \frac{1}{(2n)!} \int_o^1 \phi_{2n}(\omega) \{ \sum_{i=o}^{b-a-1} f^{(2n)}(a+\omega+i)\} \, d\omega$$

Since (B.4) with R_n given by (B.5) is an exact formula, under general conditions b can be taken equal to infinity. In many cases formula (B.4) may be used to obtain asymptotic expansions for sums (For example, Stirling's expansion for ln(n!) is immediately obtained using (B.4)).

As an example, let us use (B.4) for evaluating the function

(B.6) $$\eta(s) = \sum_{i=1}^{\infty} \ln\left(\frac{ai^{\alpha}}{ai^{\alpha}+s}\right) , \qquad \alpha > 1,$$

appearing in connection with the power rate of charge accumulation case. Using the summation method, we obtain, that for large s

(B.7) $$\eta(s) = \int_{1}^{\infty} \ln\left(\frac{au^{\alpha}}{au^{\alpha}+s}\right)du + \frac{1}{2}\ln\left(\frac{a}{a+s}\right) - R_1 =$$

$$= -\frac{\pi a^{-1/\alpha}}{\sin(\pi/\alpha)} s^{1/\alpha+\alpha} - \frac{1}{2}\ln\left(\frac{a}{a+s}\right) - R_1 + o(s^{-1/\alpha}),$$

where (after the integration by parts performed),

(B.8) $$R_1 = -\frac{1}{2!}\int_{o}^{1} (2\omega-1) \sum_{i=1}^{\infty} \frac{\alpha s}{(\omega+i)[a(\omega+i)^{\alpha}+s]} d\omega .$$

APPENDIX C

ON THE SADDLE POINT METHOD

The saddle point method is used to evaluate integrals which can be put in the form

$$F(\lambda) = \int_C \phi(z)e^{\lambda f(z)}dz \ , \tag{C.1}$$

where C is some contour in the complex plane, $\phi(z)$ and $f(z)$ are analytic functions of z in some domain of the complex plane which contains C and λ is a large positive number. (In fact, the method may be modified to include the cases when λ is complex and the functions $\phi(z)$ and $f(z)$ depend on λ. However, we shall not consider these possibilities). The saddle point method results in asymptotic approximations for $F(\lambda)$. It is based on the following theorem

<u>Theorem C.1</u> Let g and h be functions on the interval (α,β) for which the integral

$$\psi(\lambda) = \int_\alpha^\beta g(t)e^{\lambda h(t)}dt \tag{C.2}$$

exists for each sufficiently large positive λ . Let h be real, continuous at $t = \alpha$, continuously differentiable for $\alpha < t < \alpha + \eta$ $(\eta > 0)$, and such that $h' < 0$ for $\alpha < t < \alpha + \eta$. Suppose that for some $\varepsilon > 0$ and every t in $(\alpha + \eta, \beta)$

$$h(t) \leq h(a) - \varepsilon$$

holds; let $h'(t) \sim -a(t - \alpha)^{\nu-1}$ and $g(t) \sim b(t - \alpha)^{\omega-1}$ as $t \downarrow \alpha$, $\omega > 0$, $\nu > 0$. Then as $\lambda \to \infty$,

$$\text{(C.3)} \qquad \psi(\lambda) \sim \frac{b}{\nu} \cdot \Gamma\left(\frac{\omega}{\nu}\right) \left(\frac{\nu}{a\lambda}\right)^{\omega/\nu} \cdot e^{\lambda h(\alpha)}$$

<u>Proof</u>: The proof is based on the Taylor's expansion of $h(t)$ at the point $t = \alpha$ in (C.2). Details may be found in (Erdelyi (1956, p. 37)).

□

<u>Remark</u> Note that $g(t)$ may be a complex function of t.

Now let us return to the integral (C.1). The essence of the saddle point method is in the fact that for large λ the value of $F(\lambda)$ is determined by the part of the contour C on which $|e^{\lambda f(z)}|$ is large relatively to other parts of C. Since

$$\text{(C.4)} \qquad |e^{\lambda f(z)}| = e^{\lambda \mathrm{Re} f(z)} \quad ,$$

of interest is the part of C on which $\mathrm{Re} f(z)$ is large relatively to other parts. We can see that the shorter is this part and the steeper $\mathrm{Re} f(z)$ decreases in the region of the appropriate point of maximum the better $F(\lambda)$ is evaluated by only using the values of the integrand in this region (in the sense that a small relative error may be attained for

moderate values of λ). In accordance with the stated facts we try to deform C into a more appropriate path $\tilde{C}$ (by the well known Cauchy theorem, if C and $\tilde{C}$ have common endpoints and no singular point is crossed during the deformation, the value of integral remains unchanged). To explain the choice of $\tilde{C}$ we first recall the following facts (the proofs may be found in Titchmarsh (1964)).

Statement C.1 The functions Ref(z) and Im f(z) are harmonic as functions of x and y in the s-plane ($z = x+iy$), so they cannot have extremal points. The points in which $f'(z) = 0$ are saddle points of both of these functions.

□

Remark: f(x,y) is called harmonic in D if it satisfies the relation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \equiv 0 \quad , \quad (x,y) \in D.$$

Statement C.2 The contour lines of the functions Re f(z) and Im f(z) as functions of x and y form an orthogonal net.

□

It is clear from the last statement how the saddle point method operates. First of all we find the points for which $f'(z) = 0$. They are called saddle points (or cols). Suppose z_o is a unique saddle point and we are able to transform the contour in such a way that it passes through the saddle point

and coincides with the line $\operatorname{Im} f(z) = \text{const}$ passing through z_o (in the direction corresponding to maximum of $\operatorname{Re} f(z)$ in z_o) excepting the part on which the integral may be evaluated and shown to be negligibly small. (It can be seen from the Statements C1-C2 that the path of the "steepest descent" for $\operatorname{Re} f(z)$ at any point must coincide with the contour line of $\operatorname{Im} f(z)$ passing through this point. Another reason for choosing the new path as stated will become clear from (C.5) below). Suppose z_o is the only point on the deformed contour in which $\operatorname{Re} f(z)$ reaches its maximum. Let $\{z = z(t),\ t_1 \leq t \leq t_2\}$ be a parametrization of $\tilde{C}$ the point t_o corresponding to z_o. Then

$$F(\lambda) = \int_{t_1}^{t_2} \phi(z(t))e^{\lambda \operatorname{Re} f(z(t)) + i\lambda \operatorname{Im} f(z(t))} z'(t)dt, \tag{C.5}$$

Since $\operatorname{Im} f(z(t)) \equiv \text{const}$, the oscillatory behaviour of the integrand is neutralized. After we denote

$$\begin{aligned} g(t) &= \phi(z(t))z'(t) \\ h(t) &= \operatorname{Re} f(z(t)) \end{aligned} \tag{C.6}$$

and split the integral in (C.5) into two parts

$$\int_{t_1}^{t_2} = \int_{t_o}^{t_2} - \int_{t_o}^{t_1} \tag{C.7}$$

Theorem C.1 may be applied under fairly general conditions to both integrals resulting in an asymptotic representation for $F(\lambda)$.

The saddle point method may succeed in much more complicated situations (for example, when several saddle points are present, when singular points of the integrand must be crossed to obtain a new contour and the theory of residues must be used etc.). Many examples of its application may be found in Erdelyi (1956), de Bruijn (1970), Dingle (1973) and other works on asymptotic analysis.

APPENDIX D

THE CLASSIFICATION OF BIRTH-DEATH PROCESSES

Consider a birth-death process with transition probability matrix $P(t) = \{P_{ij}(t)\}$. This matrix satisfies the differential equations

(D.1) $$P'(t) = AP(t) \quad \text{(backward equations)}$$

(D.2) $$P'(t) = P(t)A \quad \text{(forward equations)},$$

the initial condition

(D.3) $$P(0) = I$$

and has the following properties

(D.4) $$P_{ij}(t) \geq 0$$

(D.5) $$\sum_{j=o}^{\infty} P_{ij}(t) \leq 1$$

(D.6) $$P(t+s) = P(t)P(s).$$

The matrix $A = \{a_{ij}\}$ is of the form

$$
\text{(D.7)} \qquad \begin{aligned}
&a_{o,o} = -\lambda_o \ , \quad a_{i,i} = -(\lambda_i + \mu_i) \ , \quad i = 1,2,\ldots \ , \\
&a_{i,i+1} = \lambda_i \ , \quad i = 0,1,\ldots \ , \\
&a_{i,i-1} = \mu_i \ , \quad i = 1,2,\ldots \ ,
\end{aligned}
$$

where $\lambda_i > 0$ for $i \geq 0$, $\mu_i > 0$ for $i \geq 1$.

Let us classify the birth-death processes in accordance with the birth rates λ_i and death rates μ_i. First of all, let us define the potentials of states by

$$
\text{(D.8)} \qquad \pi_o = 1 \ ; \quad \pi_i = \frac{\lambda_o \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} \ , \quad i \geq 1.
$$

It is known (See Karlin and McGregor (1957a, p. 527)) that there is one and only one matrix $P(t)$ satisfying (D.1)-(D.6) if and only if

$$
\text{(D.9)} \qquad \sum_{i=o}^{\infty} \left(\pi_i + \frac{1}{\lambda_i \pi_i}\right) = \infty
$$

and this is supposed to hold in what follows.

Consider a specified state i. This state is called <u>recurrent</u> if, started at i the process will return to i in finite time with probability 1. Otherwise the state is called <u>transient</u>. A recurrent state i is called <u>ergodic</u> or <u>recurrent null</u> according as its expected recurrence time is finite

or infinite. A process is called <u>recurrent</u>, <u>ergodic</u>, <u>recurrent null</u>, or <u>transient</u> if every one of its states has the corresponding property.

The following theorems identify the type of birth-death process (proofs may be found in Karlin and McGregor (1957b)).

<u>Theorem D.1</u> The process is recurrent if and only if

$$\sum_{i=o}^{\infty} \frac{1}{\lambda_i \pi_i} = \infty \tag{D.10}$$

□

<u>Theorem D.2</u> a) The process is ergodic if and only if

$$\sum_{i=o}^{\infty} \frac{1}{\lambda_i \pi_i} = \infty \quad ; \quad \sum_{i=o}^{\infty} \pi_i < \infty \tag{D.11}$$

b) The process is recurrent null if and only if

$$\sum_{i=o}^{\infty} \frac{1}{\lambda_i \pi_i} = \infty \quad ; \quad \sum_{i=o}^{\infty} \pi_i = \infty \tag{D.12}$$

□

<u>Theorem D.3</u> The process is transient if and only if

(D.13) $$\sum_{i=o}^{\infty} \frac{1}{\lambda_i \pi_i} < \infty \quad ; \quad \sum_{i=o}^{\infty} \pi_i = \infty \quad *)$$

□

Now consider transient processes only. Denote by T the time at which the particle reaches infinity when the initial state is i.

<u>Theorem D.4</u> If $P\{T < \infty\} > 0$, then $P\{T < \infty\} = 1$ and the function $(1 - H_i(t))$

$$1 - H_i(t) = P\{T > t\} = \sum_{j=o}^{\infty} P_{ij}(t)$$

tends to zero exponentially fast as $t \to \infty$.

□

There are two types of transient processes. A transient process is called of type 1 if for some finite $t > 0$ and for some i

(D.14) $$\sum_{j=o}^{\infty} P_{ij}(t) < 1 \quad (\text{i.e. } P\{T < \infty\} = 1)$$

---

*) We can see that the possibility

$$\sum_{i=o}^{\infty} \frac{1}{\lambda_i \pi_i} < \infty \quad ; \quad \sum_{i=o}^{\infty} \pi_i < \infty$$

contradicts (D.9)

and is called of type 2 if

$$\text{(D.15)} \qquad \sum_{j=o}^{\infty} P_{ij}(t) = 1 \quad (\text{i.e. } P\{T < \infty\} = 0)$$

for all i and all $t > 0$.

Before giving the characterization for both types let us prove the following statement.

<u>Statement D.1</u> If (D.14) hold for some finite $t^* > 0$ and for some i^* then it holds for every i and every $t > 0$.

<u>Proof</u>: Let us fix a point t ($t > t^*$). Then

$$\text{(D.16)} \qquad \sum_{j=o}^{\infty} P_{ij}(t) < 1$$

for every i. (Because for every i $P_{ii^*}(t-t^*) > 0$).
Let us show that (D.16) implies that

$$\text{(D.17)} \qquad \sum_{j=o}^{\infty} P_{ij}(\tfrac{3t}{4}) < 1$$

for every i. Suppose (D.17) is not true for some state $\tilde{i}$. Then

$$\text{(D.18)} \qquad \sum_{j=o}^{\infty} P_{ij}(\tfrac{t}{2}) = 1$$

for every i. But now, by semigroup property of P(t), for every i

$$\text{(D.19)} \qquad \sum_{j=o}^{\infty} P_{ij}(t) = \sum_{j=o}^{\infty} \{ \sum_{k=o}^{\infty} P_{ik}(t/2)P_{kj}(t/2)\} = 1$$

and this contradicts to (D.16). Since (D.16) implies that for every $a > 0$ and i

$$\text{(D.20)} \qquad \sum_{j=o}^{\infty} P_{ij}(t+a) < 1 ,$$

the statement is proved.

□

Now let us characterize the both types of transient processes.

<u>Theorem D.5</u> If the process is transient the following conditions are equivalent

1) The process is of type 1, that is (D.16) hold for every state i and $t > 0$.

2) It holds

$$\text{(D.21)} \qquad \sum_{i=o}^{\infty} \pi_i \sum_{j=i}^{\infty} \frac{1}{\lambda_j \pi_j} = \infty \qquad *)$$

□

---

*) The left hand side of (D.21) has a simple probabilistic meaning - it is equal to ET for initial state 0. The term $\pi_i \sum_{k=o}^{\infty} \frac{1}{\lambda_j \pi_j}$ represents the mean time the particle spent in state i.

APPENDIX E

CHOICE OF THE APPROPRIATE FUNCTION $\theta(\gamma,N,\beta)$ WHEN FINDING THE NORMALIZING CONSTANTS IN THE CASE $\lambda_j = a(j+1)^{\alpha}$

Substituting (5.2.23) into (5.2.20) we obtain

$$\text{(E.1)} \quad A(t) = \lim_{N\to\infty} NF(b_N t + a_N) =$$

$$= \lim_{N\to\infty} \left[\frac{t}{\beta(\ln N)^{(\beta+1)/\beta}} + \frac{1}{(\ln N)^{1/\beta}} + \theta\right]^{\gamma} \exp\{\ln N \cdot [1-(\frac{t}{\beta \ln N} + \theta \cdot (\ln N)^{\frac{1}{\beta}} + 1)^{-\beta}]\}$$

Here we shall impose the first condition on θ :

$$\text{(E.2)} \qquad \lim_{N\to\infty} \theta \cdot (\ln N)^{1/\beta} = 0$$

We know also that $\lim_{x\to o} \frac{1}{x}[(x+1)^r - 1] = r$, for every r.

So, A(t) can be put into the following form:

$$\text{(E.3)} \quad A(t) = \lim_{N\to\infty} (\ln N)^{-\frac{\gamma}{\beta}} \cdot \exp\left\{\frac{1-\left(\frac{t}{\beta \ln N} + \theta(\ln N)^{\frac{1}{\beta}}+1\right)^{-\beta}}{\left(\frac{t}{\beta \ln N} + \theta(\ln N)^{1/\beta}\right)} \cdot \frac{\ln N}{\left(\frac{t}{\beta \ln N} + \theta(\ln N)^{1/\beta}\right)^{-1}}\right\}$$

$$= e^{t} \cdot \lim_{N\to\infty} \{\exp[-\frac{\gamma}{\beta} \ln(\ln N)]\} \cdot \{\exp[\beta \cdot \theta \cdot (\ln N)^{(\beta+1)/\beta}]\}$$

and, since we are looking for θ that gives $\lim_{N\to\infty} A(t) = e^{t}$, it can be chosen from the equation

$$\text{(E.4)} \qquad -\frac{\gamma}{\beta} \ln(\ln N) + \beta \cdot \theta \cdot (\ln N)^{(\beta+1)/\beta} = 0 ,$$

which coincides with (5.2.24).

APPENDIX F

EIGENVALUES AND EIGENVECTORS OF THE INFINITESIMAL MATRIX A

Consider the infinitesimal matrix A (defined by (2.2.7)) corresponding to the birth-death process with absorbing state M. Let the polynomials $\{Q_o(x), Q_1(x), \ldots, Q_{M-1}(x)\}$ be defined by relations (2.2.11). We know, that the only eigenvalues of A are the solutions $\{x_o, x_1, \ldots, x_{M-1}\}$ of (2.2.12) taken with negative sign and the right eigenvector, corresponding to eigenvalue $(-x_i)$ is given by

$$\text{(F.1)} \qquad \vec{R}_i = \{Q_o(x_i), Q_1(x_i), \ldots, Q_{M-1}(x_i)\} \quad , \quad i = 0,1,\ldots,M-1.$$

Let us prove a statement concerning the <u>left</u> eigenvectors of A.

<u>Statement F.1</u> The <u>left</u> eigenvector, corresponding to the eigenvalue $(-x_i)$ is given by

$$\text{(F.2)} \quad \vec{L}_i = \{\pi_o Q_o(x_i), \pi_1 Q_1(x_o), \ldots, \pi_{M-1} Q_{M-1}(x_i)\}, \quad i = 0,1,\ldots,M-1,$$

where $\{\pi_o, \pi_1, \ldots, \pi_{M-1}\}$ are defined by (4.2.2).

<u>Proof</u>: The proof is straightforward. The $k^{\underline{th}}$ component of $\vec{L}_i A$ $(k = 1,2,\ldots,M-2)$ is given by

$$(F.3)\quad (\vec{L}_i A)_k = \lambda_{k-1}\pi_{k-1}Q_{k-1}(x_i)-(\lambda_k+\mu_k)\pi_k Q_k(x_i)+\mu_{k+1}\pi_{k+1}Q_{k+1}(x_i) =$$

$$= \pi_k\{\mu_k Q_{k-1}(x_i)-(\lambda_k+\mu_k)Q_k(x_i)+\lambda_k Q_{k+1}(x_i)\} =$$

$$= -x_i\pi_k Q_k(x_i) = (-x_i)(\vec{L}_i)_k \quad ,$$

by the definition of the polynomials (2.2.11). We can see, that the equality is preserved also for the zero$\underline{\text{th}}$ component:

$$(F.4)\quad (\vec{L}_i A)_o = -\lambda_o\pi_o Q_o(x_i) + \mu_1\pi_1 Q_1(x_i) = -x_i\pi_o Q_o(x_i) = (-x_i)(\vec{L}_i)_o \quad .$$

Concerning the last component we have

$$(F.5)\quad (\vec{L}_i A)_{M-1} = \lambda_{M-2}\pi_{M-2}Q_{M-2}(x_i) - (\lambda_{M-1}+\mu_{M-1})\pi_{M-1}Q_{M-1}(x_i) =$$

$$= \pi_{M-1}\{\mu_{M-1}Q_{M-2}(x_i) - (\lambda_{M-1}+\mu_{M-1})Q_{M-1}(x_i)\} =$$

$$= (-x_i)\pi_{M-1}Q_{M-1}(x_i) = (-x_i)(\vec{L}_i)_{M-1} \quad ,$$

since x_i is the root of (2.2.12).

So, the statement is proved.

□

From the last statement an important corollary follows:

<u>Corollary F.1</u> If x_i is a root of (2.2.12) (i.e. $(-x_i)$ is a spectral value of A) then

$$(-x_i) \sum_{j=o}^{M-1} \pi_j Q_j(x_i) = -\lambda_{M-1}\pi_{M-1}Q_{M-1}(x_i) \tag{F.6}$$

<u>Proof</u>: After we sum the equalities (F.3), (F.4) and (F.5), the result (F.6) follows.

APPENDIX G

TRANSFORMING THE CONTOUR OF INTEGRATION AND MAPPING IN THE INTEGRAL FORM OF F(t) IN THE CASE $\lambda_j = b\lambda^j$.

Let us consider the expression (5.2.53) for F(t/b). The contour of integration G-ζ starts at $(\infty - i\pi/2)$, crosses the real axis in the origin and turns back to $(\infty + i\pi/2)$. G-ζ lies entirely in the strip $|\mathrm{Im}\ z| < \pi/2$.

First of all let us show that this contour can be replaced by a new one (denote it by V) without affecting the value of the integral. The contour V is defined as follows: it starts at $(\infty - i\pi)$, crosses the origin, turns back to $(\infty + i\pi)$ and lies entirely in the strip $|\mathrm{Im}\ z| < \pi$. We can see that

$$\text{(G.1)} \qquad \int_{G-\zeta} = \int_V + \int_{\infty - \frac{i\pi}{2}}^{\infty - i\pi} - \int_{\infty + \frac{i\pi}{2}}^{\infty + i\pi} .$$

But, since $|c(s)|$ is bounded for large s and for $1 < x < 2$

$$\text{(G.2)} \qquad \left|\exp\left\{-\frac{1}{2a}\left[K \pm \frac{i\pi x}{2}\right]^2 + \frac{q}{a}\left[e^{K \pm \frac{i\pi x}{2}} - \left(K \pm \frac{i\pi x}{2}\right) - 1\right]\right\}\right| =$$

$$= \exp\left\{-\frac{1}{2a}K^2 + \frac{\pi^2 x^2}{8a} + \frac{q}{a}\left[e^K \cos\frac{\pi x}{2} - K - 1\right]\right\} \to 0$$

as $K \to \infty$, the integrals $\int_{\infty \pm \frac{i\pi}{2}}^{\infty \pm i\pi}$ appearing in (G.1) are equal to zero.

Thus, the integration in (5.2.53) can be performed along the path V.

Now let us consider the mapping $\xi = (2(e^{\omega} - \omega - 1))^{1/2}$, where this branch of the square root is used for which $(1)^{1/2} = 1$. Since

$$(G.3) \qquad [2(e^{K\pm i\pi} - (K\pm i\pi) - 1)]^{1/2} \to \pm i\infty \qquad \text{as } K \to \infty ,$$

the point $\omega = 0$ corresponds to $\xi = 0$ and the point $\omega = 0$ is a single zero of the mapping, the path V is transformed by the mapping into the imaginary axis in ξ-plane. This leads immediately to (5.2.55).

APPENDIX H

ON OSCILLATORY BEHAVIOUR OF $\phi(q)$ IN THE CASE $\lambda_j = b\lambda^j$.

We obtained the following expression for $\phi(q)$ (See 5.2.66):

$$\text{(H.1)} \quad \phi(q) = \psi(\alpha) = \frac{a^2}{2}\int_0^1 (\omega^2-\omega) \sum_{j=-\infty}^{\infty} \frac{\exp[a(\omega+j+\alpha)]}{\{1+\exp[a(\omega+j+\alpha)]\}^2}\, d\omega =$$

$$= \frac{a}{2}\sum_{j=-\infty}^{\infty}\int_0^1 (2\omega-1)\cdot\frac{1}{1+\exp[a(\omega+j+\alpha)]}\, d\omega \quad ,$$

where α is 1/2 minus the fractional part of q/a. Let us show that $\phi(q)$ does not approach a limit as $q \to \infty$ (i.e. that $\psi(\alpha)$ depends on α). Since

$$\text{(H.2)} \quad \psi'(\alpha) = \frac{a}{2}\sum_{j=-\infty}^{\infty} (2\omega-1)\, d\left[\frac{1}{1+\exp[a(\omega+j+\alpha)]}\right] =$$

$$= \frac{a}{2}\sum_{j=-\infty}^{\infty}\left\{\frac{1}{1+\exp[a(1+j+\alpha)]} + \frac{1}{1+\exp[a(j+\alpha)]} - 2\int_0^1 \frac{d\omega}{1+\exp[a(\omega+j+\alpha)]}\right\} =$$

$$= \frac{a}{2}\sum_{j=-\infty}^{\infty}\left\{-\frac{\exp[a(1+j+\alpha)}{1+\exp[a(1+j+\alpha)]} - \frac{\exp[a(j+\alpha)]}{1+\exp[a(j+\alpha)]} + \frac{2}{a}\ln\frac{1+\exp[a(1+j+\alpha)]}{1+\exp[a(j+\alpha)]}\right\} =$$

$$= \frac{a}{2}\left\{\alpha - \frac{1}{2} + \frac{1}{1+\exp(a\alpha)} + \sum_{j=1}^{\infty}\left[\frac{1}{1+\exp[a(j+\alpha)]} - \frac{1}{1+\exp[a(j-\alpha)]}\right]\right\}$$

We see now that $\psi'(\alpha)$ is not identically equal to zero by considering large values of a. However, for moderate values of a $\psi'(\alpha)$ is fairly close to zero (See Table H.1). It can be shown that the only zeros of $\psi'(\alpha)$ in the interval $(-\frac{1}{2}, \frac{1}{2}]$ are $\alpha_1 = 0$ and $\alpha_2 = 1/2$. Since $\psi(\alpha)$ is periodic with period 1, the last fact means that one of the points α_1, α_2 corresponds to maximum and another one to minimum of $\psi(\alpha)$. So, the range of $\psi(\alpha)$ is equal to

$$\max_{\alpha} \psi(\alpha) - \min_{\alpha} \psi(\alpha) = |\psi(\tfrac{1}{2}) - \psi(0)| \tag{H.3}$$

The last equality estimates the amplitude of oscillations of the function $c(\frac{q}{at})$ appearing in the asymptotic representation of the distribution function (See 5.2.58).

Table H.1 Dependence of $\psi'(\alpha)$ on α

a = ln 2		a = ln 4	
α	$\psi'(\alpha)$	α	$\psi'(\alpha)$
-0.5	0	-0.5	0
-0.4	0.16×10^{-11}	-0.4	0.24×10^{-5}
-0.3	0.26×10^{-11}	-0.3	0.39×10^{-5}
-0.2	0.26×10^{-11}	-0.2	0.39×10^{-5}
-0.1	0.16×10^{-11}	-0.1	0.24×10^{-5}
0.0	0	0.0	0
0.1	-0.16×10^{-11}	0.1	-0.24×10^{-5}
0.2	-0.26×10^{-11}	0.2	-0.39×10^{-5}
0.3	-0.26×10^{-11}	0.3	-0.39×10^{-5}
0.4	-0.16×10^{-11}	0.4	-0.24×10^{-5}
0.5	0	0.5	0

APPENDIX I

Justification of the termwise passage to the limit for finding $\tilde{k}$ in the case of power charge accumulation rates.

To justify the termwise passage to the limit it is enough to show that $\sum_{i=1}^{\infty} A_i \to 0$ as $s\to\infty$, where

$$(I.1)\quad A_i = \left|\frac{\alpha}{2}\int_0^1 \frac{(2w-1)s\,dw}{(w+i)\left[a(w+i)^{\alpha}+s\right]} - \frac{\alpha}{2}\int_0^1 \frac{(2w-1)}{(w+i)}\,dw\right| =$$

$$= \left|\int_0^1 \frac{(2w-1)}{2}\,d\{\ln\left[a(w+i)^{\alpha}+s\right]\}\right| =$$

$$= \left|\frac{\ln\left[a(i+1)^{\alpha}+s\right]+\ln\left[ai^{\alpha}+s\right]}{2} - \int_i^{i+1} \ln\left[az^{\alpha}+s\right]dz\right|$$

(the last expression is obtained by integration by parts). The function $\ln\left[az^{\alpha}+s\right]$ is convex for $z<\left[(\alpha-1)s/a\right]^{1/\alpha}$, concave for $z>\left[(\alpha-1)s/a\right]^{1/\alpha}$ and monotone. Since the existence of $\tilde{k}$ follows from the existence of the asymptotic expansion for $\sum_{j=0}^{\infty}(aj^{\alpha}+s)^{-1}$, any subsequence $\{s_k\}_{k\to\infty}^{\to\infty}$ may be used to find $\tilde{k}$ by means of (5.2.3). Let us take a subsequence $\{S_k\}$ for which $\left|(\alpha-1)s_k/a\right|^{1/\alpha}=k$.

Since for a function $f(x)$, convex (or concave) and continuous in $[0,1]$ the inequality

$$(I.2)\quad \left|\frac{f(1)+f(0)}{2} - \int_0^1 f(x)\,dx\right| \le \left|\frac{f(1)+f(0)}{2} - f(\tfrac{1}{2})\right|$$

holds (it follows from Jensen's inequality), for any integers i and k

$$(I.3)\quad A_i \le \left|\ln\{1+\frac{\left[a(i+1)^{\alpha}+s_k\right]\left[ai^{\alpha}+s_k\right]-\left[a(i+\frac{1}{2})^{\alpha}+s_k\right]^2}{\left[a(i+\frac{1}{2})^{\alpha}+s_k\right]^2}\}\right|$$

Furthermore, denoting the fraction in the last expression by F_i,

$$\text{(I.4)} \quad |F_i| = \left| \frac{a^2\{[i(i+1)]^\alpha - [(i+\tfrac{1}{2})^2]^\alpha\} + as_k\{(i+1)^\alpha + i^\alpha - 2(i+\tfrac{1}{2})^\alpha\}}{[a(i+\tfrac{1}{2})^\alpha + s_k]^2} \right| \le$$

$$\le c_1 \left| \frac{(i+\tfrac{1}{2})^{2\alpha-2}}{[a(i+\tfrac{1}{2})^\alpha + s_k]^2} \right| + c_2 \left| \frac{s_k(i+C)^{\alpha-2}}{[a(i+\tfrac{1}{2})^\alpha + s_k]^2} \right|$$

(Here, $c_1 > 0$, $c_2 > 0$, $0 < C < 1$ are positive constants. The first bound is obtained using the inequality $x^\alpha - y^\alpha \le rx^{\alpha-1}(x-y)$ for $x \ge y \ge 0$ (see Hardy (1973, p. 39)) and the second one by using the Taylor expansion of $[(i+1)^\alpha + i^\alpha]$ about the point $(i+\tfrac{1}{2})$).

Now, using the inequality of the arithmetic and geometric means (see Hardy (1973, p.17)) we obtain that for every $0 < \beta < 1$

$$\text{(I.5)} \quad [a(i+\tfrac{1}{2})^\alpha + s_k] \ge C(i+\tfrac{1}{2})^{\alpha\beta} s_k^{1-\beta} ,$$

where C is a positive constant, dependent on a and β only. Choosing β appropriately, we obtain bounds for both terms in the right hand side of (I.4). Thus, for every i and k

$$\text{(I.6)} \quad |F_i| \le C_1(i+\tfrac{1}{2})^{-1-\varepsilon_1} s_k^{-\varepsilon_2} + C_2(i+\tfrac{1}{2})^{-1-\varepsilon_3} s_k^{-\varepsilon_4} ,$$

where $C_1, C_2, \varepsilon_1, \ldots, \varepsilon_4$ are positive constants. The last relation implies that $\sum_{i=1}^{\infty} A_i \to 0$ as $s \to \infty$.

APPENDIX J

Properties of the estimators $\widehat{1/\gamma}$ and $(\widehat{\ln b_N})$

In this appendix we display the calculated values of the bias and standard deviation for the estimators $\widehat{1/\gamma}$ and $(\widehat{\ln b_N})$ (see (3.3.27) and (3.3.56) for various values of γ and k. As well, we give the calculated correlation coefficient between the above estimators. In tables J.1 and J.2 the upper number in each box represents the bias and the lower number represents the standard deviation.

k \ $1/\gamma$	16	8	4	2	1	0.5	0.25	0.125	0.0625
4	0.101	0.192	0.343	0.545	0.750	0.908	1.008	1.064	1.094
	11.249	5.548	2.688	1.345	0.909	0.891	0.949	1.013	1.131
5	0.068	0.134	0.252	0.424	0.611	0.763	0.861	0.916	0.946
	9.201	4.553	2.215	1.095	0.702	0.668	0.708	0.753	0.835
6	0.051	0.102	0.198	0.347	0.521	0.667	0.763	0.818	0.847
	7.976	3.955	1.931	0.949	0.587	0.545	0.576	0.611	0.674
7	0.041	0.083	0.162	0.295	0.457	0.598	0.692	0.747	0.776
	7.138	3.545	1.736	0.851	0.514	0.467	0.492	0.521	0.571
8	0.034	0.069	0.138	0.256	0.408	0.545	0.639	0.693	0.722
	6.519	3.241	1.591	0.780	0.462	0.413	0.433	0.458	0.499
9	0.029	0.059	0.119	0.227	0.370	0.503	0.596	0.650	0.679
	6.037	3.004	1.478	0.724	0.423	0.373	0.390	0.411	0.447
10	0.026	0.052	0.105	0.203	0.340	0.469	0.561	0.614	0.643
	5.649	2.812	1.386	0.679	0.392	0.342	0.357	0.376	0.406
15	0.016	0.032	0.066	0.134	0.245	0.361	0.448	0.501	0.529
	4.434	2.212	1.097	0.541	0.304	0.256	0.263	0.276	0.204
20	0.011	0.023	0.048	0.100	0.194	0.302	0.385	0.437	0.466
	3.770	1.883	0.937	0.466	0.261	0.216	0.221	0.231	0.243
25	0.009	0.018	0.037	0.080	0.162	0.263	0.344	0.395	0.423
	3.336	1.667	0.833	0.418	0.236	0.196	0.199	0.207	0.216
30	0.007	0.015	0.031	0.067	0.140	0.235	0.315	0.365	0.393
	3.024	1.513	0.758	0.384	0.221	0.184	0.186	0.193	0.200

Table J.1: Bias and standard deviation of $\widehat{1/\gamma}$

k \ 1/γ	16	8	4	2	1	0.5	0.25	0.125	0.0625
4	-0.128 16.501	-0.248 8.163	-0.464 3.974	-0.806 1.972	-1.275 1.304	-1.841 1.311	-2.466 1.436	-3.124 1.546	-3.800 1.689
5	-0.103 15.770	-0.205 7.820	-0.395 3.819	-0.714 1.876	-1.170 1.184	-1.731 1.169	-2.355 1.289	-3.012 1.391	-3.688 1.517
6	-0.088 15.218	-0.176 7.558	-0.346 3.703	-0.645 1.812	-1.089 1.107	-1.644 1.076	-2.266 1.190	-2.923 1.288	-3.599 1.401
7	-0.077 14.766	-0.155 7.342	-0.309 3.605	-0.591 1.762	-1.022 1.052	-1.572 1.008	-2.193 1.118	-2.850 1.212	-3.525 1.314
8	-0.069 14.378	-0.139 7.155	-0.280 3.521	-0.546 1.721	-0.966 1.010	-1.511 0.955	-2.130 1.061	-2.787 1.152	-3.462 1.247
9	-0.063 14.039	-0.127 6.991	-0.257 3.446	-0.509 1.685	-0.918 0.976	-1.459 0.913	-2.076 1.016	-2.732 1.104	-3.407 1.193
10	-0.058 13.737	-0.117 6.844	-0.238 3.379	-0.477 1.653	-0.876 0.948	-1.412 0.878	-2.028 0.978	-2.684 1.065	-3.358 1.148
15	-0.042 12.581	-0.085 6.278	-0.176 3.116	-0.368 1.535	-0.726 0.858	-1.241 0.769	-1.850 0.858	-2.504 0.937	-3.178 1.004
20	-0.034 11.769	-0.069 5.879	-0.141 2.928	-0.303 1.454	-0.629 0.810	-1.127 0.714	-1.730 0.795	-2.383 0.870	-3.056 0.928
25	-0.029 11.149	-0.058 5.573	-0.119 2.784	-0.260 1.395	-0.561 0.784	-1.044 0.686	-1.642 0.762	-2.292 0.832	-2.966 0.884
30	-0.025 10.651	-0.050 5.328	-0.104 2.670	-0.228 1.351	-0.509 0.770	-0.979 0.675	-1.572 0.745	-2.222 0.811	-2.895 0.859

Table J.2: Bias and standard deviation of $(\widehat{\ln b_N})$

k \ $1/\gamma$	16	8	4	2	1	0.5	0.25	0.125	0.0625
4	-0.86	-0.85	-0.84	-0.84	-0.88	-0.92	-0.92	-0.92	-0.93
5	-0.88	-0.88	-0.87	-0.86	-0.89	-0.93	-0.93	-0.93	-0.94
6	-0.89	-0.89	-0.89	-0.88	-0.91	-0.94	-0.94	-0.94	-0.94
7	-0.91	-0.90	-0.90	-0.89	-0.91	-0.94	-0.95	-0.94	-0.95
8	-0.91	-0.91	-0.91	-0.90	-0.92	-0.95	-0.95	-0.95	-0.95
9	-0.92	-0.92	-0.92	-0.91	-0.93	-0.95	-0.95	-0.95	-0.95
10	-0.93	-0.93	-0.92	-0.92	-0.93	-0.95	-0.96	-0.95	-0.95
15	-0.94	-0.94	-0.94	-0.94	-0.95	-0.97	-0.97	-0.96	-0.96
20	-0.95	-0.95	-0.95	-0.95	-0.96	-0.97	-0.97	-0.97	-0.96
25	-0.96	-0.96	-0.96	-0.96	-0.96	-0.98	-0.97	-0.97	-0.96
30	-0.96	-0.96	-0.96	-0.96	-0.97	-0.98	-0.98	-0.97	-0.97

Table J.3: Correlation coefficient between $\widehat{1/\gamma}$ and $(\widehat{\ln b_N})$.

REFERENCES

1. Agarwal, V. (1974) Breakdown conduction in thin dielectric films. A bibliographical survey, Thin Solid Films, vol. 24, pp. 55-69.
2. Bromwich, T.I.J'a. (1965) Introduction to the Theory of Infinite Series. Macmillan, New York.
3. de Bruijn, N.G. (1953) The difference-differential equation $F'(x) = e^{\alpha x+\beta}F(x-1)$. Indagationes Math. 15, 449-458.
4. de Bruijn, N.G. (1970) Asymptotic Methods in Analysis. North Holland Publishing Co., Amsterdam.
5. Chung, K.L. (1967) Markov Chains with Stationary Transition Probabilities. Springer-Verlag, N.Y.
6. Chung, K.L. (1956) Foundations of the theory of continuous parameter Markov chains, Proceedings of the Third Berkeley Symposium, vol. 2, pp. 29-40.
7. Cox, D.R. and Miller, H.D. (1965) The Theory of Stochastic Processes. Methuen & Co. Ltd., London.
8. David, H.A. (1981) Order Statistics. John Wiley & Sons, N.Y.
9. De Haan, L. (1971) On regular variation and its application to the weak convergence of sample extremes. Mathematical Centre Tracts, vol. 32, Amsterdam.
10. Dingle, R.B. (1973) Asymptotic Expansions: Their Derivation and Interpretation. Academic Press, London.
11. DiStefano, T.H. and Shatzkes, M. (1976) Dielectric instability and breakdown in SiO_2 thin films, J. Vac. Sci. Technol., vol. 13, No. 1, pp. 50-55.

12. Doetsch, G. (1974) Introduction to the Theory and Application of the Laplace Transformation. Springer-Verlag, Berlin.

13. Epstein, B. (1962) Simple estimates of the parameters of exponential distributions. In the book by Sarhan, A. and Greenberg, B. "Contributions to Order Statistics", pp. 361-371.

14. Erdelyi, A. (1956) Asymptotic Expansions. Dover Publications, N.Y.

15. Feller, W. (1971) An introduction to Probability Theory and its Applications, vol. 2, Wiley, New York.

16. Galambos, J. (1978) The Asymptotic Theory of Extreme Order Statistics. Wiley, New York.

17. Gnedenko, B.V. (1943) Sur la distribution limite du terme maximum d'une serie aleatoire. Ann. Math. 44, pp. 423-453.

18. Gumbel, E.J. (1958) Statistics of Extremes. Columbia Univ. Press, N.Y.

19. Harari, E. (1978) Dielectric breakdown in electrically stressed thin films of thermal SiO_2, J. Appl. Physics, vol. 49, No. 4, pp. 2478-2489.

20. Hirschman, I.I. and Widder, D.V. (1955) The Convolution Transform. Princeton University Press, Princeton.

21. Johnson, N and Kotz, S. (1970) Continuous Univariate Distributions - 1. Houghton Mifflin Company, Boston.

22. Kadary, V. and Klein, N. (1980) Electrical breakdown during the anodic growth of tantalum pentoxide. J. of the Electrochemical Society, 127(1), pp. 139-151.

23. Karlin, S. and McGregor, J. (1957a) The differential equations of birth-and-death processes and the Stieltjes moment problem. Trans. Amer. Math. Soc. vol. 85, pp. 489-546.

24. Karlin, S. and McGregor, J. (1957b) The classification of birth and death processes. Trans. Amer. Math. Soc. vol. 86, pp. 366-400.

25. Karlin, S. and Taylor, H. (1975) A First Course in Stochastic Processes. Academic Press, N.Y.

26. Karlin, S. (1968) Total Positivity. Stanford University Press, Stanford, California.

27. Kashat, I. and Klein, N. (1977) Current runaway in insulators affected by impact ionization and drift. J. Appl. Physics.

28. Keilson, J. (1964) A review of transient behaviour in regular diffusion and birth-death processes. J. Appl. Prob. vol. 1, pp. 247-266.

29. Keilson, J. (1965) A review of transient behaviour in regular diffusion and birth-death processes. Part II. J. Appl. Prob. vol. 2, pp. 405-428.

30. Keilson, J. (1971) Log-Concavity and Log-Convexity in passage time densities of diffusion and Birth-Death Processes. J. Appl. Prob. vol. 8, pp. 391-398.

31. Keilson, J. (1979) Markov Chain Models - Rarity and Exponentiality. Springer-Verlag, New York.

32. Kingman, J.F.C. (1972) Regenerative Phenomena. Wiley, London.

33. Klein, N. (1971) Switching and breakdown in films. Thin Solid Films, vol. 7, pp. 149-177.

34. Klein, N. (1972) A theory of localized electronic breakdown in insulating films, Adv. Physics, vol. 21, pp. 605-644.

35. Klein, N. and Solomon, P. (1976) Current runaway in insulators affected by impact ionization and recombination. J. Appl. Physics, vol. 47, No. 10, pp. 4364-4372.

36. Klein, N. (1978) Electrical breakdown mechanisms in thin insulators, Thin Solid Films, vol. 50, pp. 223-232.

37. Korn, G. and Korn, T. (1968) Mathematical Handbook for Scientists and engineers. McGraw-Hill, N.Y.

38. Murray, J.D. (1974) Asymptotic Analysis. Clarendon Press, Oxford.

39. O'Dwyer, J.J. (1969) Theory of high field conduction in a dielectrics, J. Appl. Physics, vol. 40, No. 10, pp. 3887-3891.

40. O'Dwyer, J.J. (1973) The Theory of Electrical Conduction and Breakdown in Solid Dielectrics. Oxford Univ. Press, Oxford.

41. Rainville, E.D. (1960) Special Functions. Macmillan, N.Y.

42. Resnick, S.I. (1971) Tail equivalence and its applications. J. Appl. Prob. 8, pp. 136-156.

43. Ridley, B.K. (1975) Mechanism of electrical breakdown in SiO_2 films. J. Appl. Physics, vol. 43, No. 3, pp. 998-1007.

44. Rose, R.M., Shepard, L.A. and Wulff, J. (1966) The Structure and the Properties of Materials, vol. 3. Wiley, N.Y.

45. Rosenlund, S.I. (1977) Upwards Passage Times in the non-negative Birth-Death Process. Scand. J. Statist. vol. 4, pp. 90-92.

46. Solomon, P., Klein, N. and Albert, M. (1976) A statistical model for step and ramp voltage breakdown tests in thin insulators, Thin Solid Films, vol. 35, pp. 321-326.

47. Solomon, P. (1977) Breakdown in silicon oxide - A review. J. Vac. Sci. Technol., vol. 14, No. 5, pp. 1122-1130.

48. Titchmarsh, E.C. (1964) The Theory of Functions. Oxford Univ. Press, London.

49. Weissman, I. (1975) Multivariate extremal processes generated by independent non-identically distributed random variables. J. Appl. Prob., Vol. 12, pp. 477-487.

50. Weissman, I. (1978) Estimation of parameters and large quantiles based on the k largest observations. J. Amer. Statist. Ass., Vol. 73, pp. 812-815.

51. Whittaker, E.T. and Watson, G.N. (1963) A Course of Modern Analysis. Cambridge Univ. Press, Cambridge.

52. de Wit, H.J.,Wijenberg, C. and Crevecoeur, C. (1976) The dielectric breakdown of anodic aluminum oxide. J. Electrochem. Soc., Vol. 123, No. 10, pp. 1479-1486.

Additional References

53. Bhattacharya, R.N. and Rao, R.R. (1976) Normal Approximation and Asymptotic Expansions. Wiley, N.Y.

54. Darroch, J.N. and Seneta, E. (1965) On quasi-stationary distributions in absorbing discrete-time finite Markov chains. J. Appl. Prob., Vol. 2, pp. 88-100.

55. Gradsteyn, I.S., Ryzhik, I.M. (1965) Table of Integrals, Series and Products. Academic Press, N.Y.

56. Hardy, G.H., Littlewodd, J.E. and Polya, G. (1973) Inequalities. Cambridge Univ. Press, Cambridge.

57. Osburn, C.M. and Ormond, D.W. (1972) Dielectric breakdown in silicon dioxide films on silicon. J. of Electrochem. Soc., V. 119, p.591.

58. Pickands III, J. (1968) Moment convergence of sample extremes. Ann. Math. Statist., Vol. 39, pp. 881-889.

59. Rao, C.R. (1965) Linear Statistical Inference and Its Applications. Wiley, N.Y.

60. Rockette, H., Antle, C. and Klimko, L.A. (1974) Maximum likelihood estimation with the Weibull model. J. Amer. Stat. Ass., Vol. 69, pp. 246-249.

61. Weissman, I. (1981a) Confidence intervals for the threshold parameter. Commun. Statist. - Theory and Methods, A10(6), pp. 549-557.

62. Weissman, I. (1981b) Confidence intervals for the threshold parameter, Part II (unpublished).