Solutions of Polynomial Systems Derived from the Steady Cavity Flow Problem

[Extended Abstract] *

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ABSTRACT

We propose a general algorithm to approximately enumerate all solutions of a zero-dimensional polynomial system with respect to a given cost function. The algorithm is developed and is used to study a polynomial system obtained by discretizing the steady cavity flow problem in two dimensions. The key technique on which our algorithm is based is to solve polynomial optimization problems via sparse semidefinite program relaxations (SDPR) [18], which has been adopted successfully to solve reaction-diffusion boundary value problems in [11]. The cost function to be minimized is derived from discretizing the kinetic energy of the fluid. The solutions of the enumeration algorithm are shown to converge to the minimal kinetic energy solutions for SDPR of increasing order. We take advantage of Gröbner basis method to tune the performance of the algorithm, demonstrate the algorithm with SDPR of first and second order on polynomial systems for different scenarios of the cavity flow problem and succeed in approximately deriving the k smallest kinetic energy solutions. The question of whether these solutions converge to solutions of the steady cavity flow problem is discussed, and we pose a conjecture for the minimal energy solution for increasing Reynolds number.

Keywords

Steady cavity flow problem, finite difference discretization, polynomial optimization, semidefinite programming relaxation, sparsity

1. INTRODUCTION

The steady cavity flow problem is a simple model of a flow with closed streamlines and is used for examining and validating numerical solution techniques in fluid dynamics. Although it has been discussed in the literature of numerical analysis of fluid mechanics (see, e.g., [8], [3], [6], [4], [14]), it is still an interesting problem to a number of researchers for a range of Reynolds numbers. We are interested in a polynomial system derived from discretizing the steady cavity flow problem. This polynomial system, called the *discrete steady cavity flow problem*, is obtained by discretizing the cavity region, approximating the partial differential equation of the two-dimensional cavity flow problem by finite difference method, and depends on two parameters: The Reynolds number, R; and the grid discretization, N.

Our main contribution, presented in section 3 is an algorithm to approximately enumerate the solutions of the discrete cavity flow problem with respect to an objective function. The objective function is derived from discretizing the kinetic energy function of the flow. The key element of the enumeration algorithm is the sparse semidefinite program relaxation method (SDPR) [18] for solving polynomial optimization problems, whose solution is taken as the starting point for Newton's method or sequential quadratic programming. Recently, the SDPR has been successfully adopted to derive numerical solutions to a class of reaction diffusion equations [11]. We prove that the first k solutions provided by the enumeration algorithm converge to the k smallest energy solutions of the discrete cavity flow problem, in that case we apply SDPR of increasing relaxation order. Furthermore, we demonstrate this algorithm for different parameter settings of R, and we show how to use the Gröbner basis method to tune the parameters in SDPR and the performance of the enumeration algorithm. Secondly, we discuss the behavior of the minimal energy solution of the discrete steady cavity flow problem in the case that a finer grid is chosen to discretize the cavity flow problem in section 4. For small Reynolds numbers R standard grid-

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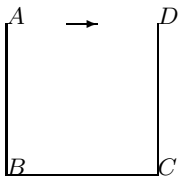
refining techniques can be applied to extend solutions of the polynomial system to finer grids. However the polynomial systems for large R behave differently and convergence is far more difficult to obtain. Therefore, we examine how the polynomial system depends on the Reynolds number R in case that the discretization N is fixed.

2. THE DISCRETE STEADY CAVITY FLOW PROBLEM

The 2-dimensional steady cavity flow problem is to solve the Navier-Stokes equation on the cavity region $ABCD$ with the coordinates $A = (0,1)$, $B = (0,0)$, $C = (1,0)$, $D = (1,1)$.



We apply the well-know stream function method (see, e.g., [8], [14]) to solve the Navier-Stokes equation numerically, and obtain the following polynomial system.

$$g^1_{i,j}(\psi,\omega) = 0 \quad \forall\, 2 \le i,j \le N-1, \tag{1}$$
$$g^2_{i,j}(\psi,\omega) = 0 \quad \forall\, 2 \le i,j \le N-1, \tag{2}$$

$$\begin{array}{ll}
\psi_{1,j} = \psi_{N,j} = 0 & \forall\, j \in \{1,\dots,N\}, \\
\psi_{i,1} = \psi_{i,N} = 0 & \forall\, i \in \{1,\dots,N\}, \\
\omega_{1,j} = -2\frac{\psi_{2,j}}{h^2} & \forall\, j \in \{1,\dots,N\}, \\
\omega_{N,j} = -2\frac{\psi_{N-1,j}}{h^2} & \forall\, j \in \{1,\dots,N\}, \\
\omega_{i,1} = -2\frac{\psi_{i,2}}{h^2} & \forall\, i \in \{1,\dots,N\}, \\
\omega_{i,N} = -2\frac{\psi_{i,N-1}+h}{h^2} & \forall\, i \in \{1,\dots,N\},
\end{array} \tag{3}$$

where

$$\begin{array}{rl}
g^1_{i,j}(\psi,\omega) = & -4\omega_{i,j} + \omega_{i+1,j} + \omega_{i-1,j} + \omega_{i,j+1} + \omega_{i,j-1} \\
& +\frac{R}{4}(\psi_{i+1,j} - \psi_{i-1,j})(\omega_{i,j+1} - \omega_{i,j-1}) \\
& -\frac{R}{4}(\psi_{i,j+1} - \psi_{i,j-1})(\omega_{i+1,j} - \omega_{i-1,j}), \\
g^2_{i,j}(\psi,\omega) = & -4\psi_{i,j} + \psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} \\
& +\psi_{i,j-1} + h^2\omega_{i,j},
\end{array}$$

and $h = 1/(N-1)$. We will call the polynomial system (1), (2), (3) the *discrete steady cavity flow problem* denoted as DSCF(R,N). It depends on two parameters, the **Reynolds number** R and the **discretization** N of the cavity region $ABCD = [0,1]^2$. Let the number $2(N-2)^2$ of variables in DSCF(R,N) corresponding to function evaluations at interior grid points, be called the **dimension n of the discrete steady cavity flow problem**. In order to visualize a numerical solution $(\psi,\omega)(N)$ we display the horizontal and vertical velocities v_1 and v_2 at $(x,y) = ((i-1)h,(j-1)h)$ given by

$$v_1 = \frac{\partial\psi}{\partial y} \approx \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2h},\ v_2 = -\frac{\partial\psi}{\partial x} \approx -\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2h}. \tag{4}$$

Moreover, a solution $(\psi,\omega)(N)$ of the discrete cavity flow problem of discretization N, one that does not converge to a physical solution of the original continuous cavity flow problem for $N \to \infty$, is called a **fake solution**. Several methods have been used to solve the cavity flow problem and the steady cavity flow problem numerically (see, e.g., [1], [3], [4], [6], [8], [14], [16]). In this paper, we propose a new method to solve the discrete steady cavity flow problem. It provides solutions sorted by their (discretized) kinetic energy.

REMARK 1. *In (1), we discretize the Jacobian* $\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}$ *by the central difference scheme. It is shown by Arakawa [1] that the central difference scheme is the simplest, but the discretized system does not keep important physical invariants. We study the system DSCF(R, N) with standard central difference scheme as the simplest starting test case.*

REMARK 2. *We conjecture that the discrete cavity flow problem DSCF(R, N) has finite complex solutions. In other words, it defines a zero-dimensional ideal. We have checked the conjecture up to* $N = 5$ *by Gröbner basis computation.*

3. SPARSE SDP RELAXATION METHOD

The main contribution of this paper is to propose an algorithm that enumerates the smallest kinetic energy solutions of the discrete steady cavity flow problem DSCF(R,N) starting with the minimal energy solution. The key element of this algorithm is to apply the *sparse semidefinite program relaxation method* (SDPR) to solve the DSCF(R,N). The SDPR for PDEs was proposed in [11] and is based on the idea to take the polynomial system derived from a finite difference discretization of a differential equation and its boundary conditions (for instance: DSCF(R,N)) as constraints for an optimization problem. Choosing another polynomial function F as objective, a polynomial optimization problem (POP) of the form

$$\begin{array}{lll}
\min & F(x) & \\
\text{s.t.} & g_j(x) \ge 0 & \forall\, j \in \{1,\dots,k\}, \\
& h_i(x) = 0 & \forall\, i \in \{1,\dots,l\}.
\end{array} \tag{5}$$

is obtained. As shown in [11], polynomial optimization problems derived from differential equations satisfy structured sparsity patterns and the sparse SDP relaxations due to [18] can be applied to approximate the solution of POP (5). The crucial point is how to choose the objective function F in POP (5). In the case that several solutions to a discretized PDE problem exist, the choice of the objective function allows to select solutions of particular interest. For the cavity flow problem, we are interested in the solution which minimizes the kinetic energy (6) given by

$$\int\int_{ABCD} \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{-\partial\psi}{\partial x}\right)^2 dxdy \tag{6}$$

Thus, for the cavity flow problem we yield by discretizing (6) the following function F as a canonical choice for the objective function of (5):

$$\begin{array}{rl}
F(\psi,\omega) = \frac{1}{4}\displaystyle\sum_{2\le i,j\le N-1} & \psi^2_{i+1,j} + \psi^2_{i-1,j} + \psi^2_{i,j+1} + \psi^2_{i,j-1} \\
& -2\psi_{i+1,j}\psi_{i-1,j} - 2\psi_{i,j+1}\psi_{i,j-1}.
\end{array} \tag{7}$$

Taking DSCF(R,N) as a system of constraints and F as objective function, we derive the polynomial optimization

problem,

$$\begin{array}{ll} \min & F(\psi,\omega) \\ \text{s.t.} & (1),\ (2),\ (3) \end{array} \tag{8}$$

We call POP (8) the **steady cavity flow optimization problem CF**(R,N) . As all polynomials in (8) are of degree at most two, CF(R,N) is a **quadratic optimization problem (QOP)**. In fact, a further classification is possible for $R=0$ and $R\neq 0$.

PROPOSITION 1. *a) CF$(0,N)$ is a **convex quadratic program** for any N.*

*b) CF(R,N) is **non-convex** for any N, if $R\neq 0$.*

Proof:

a) In case $R=0$ all constraints are linear. Furthermore, the objective function can be written as $F=\sum_{i,j} F^1_{i,j}+F^2_{i,j}$, where

$$F^1_{i,j}(\psi,\omega)=\begin{pmatrix}\psi_{i-1,j}\\ \psi_{i+1,j}\end{pmatrix}^T\begin{pmatrix}2&-2\\-2&2\end{pmatrix}\begin{pmatrix}\psi_{i-1,j}\\ \psi_{i+1,j}\end{pmatrix}.$$

It follows that $F^1_{i,j}$ is convex as $\begin{pmatrix}2&-2\\-2&2\end{pmatrix}$ positive semidefinite with eigenvalues 0 and 4. The convexity of $F^2_{i,j}$ follows analoguously. Thus, F can be written as a sum of convex function and is therefore convex as well. The proposition follows.

b) In case $R\neq 0$, the equality constraint function $g^1_{i,j}$ is indefinite quadratic. Thus, CF(R,N) is a non-convex quadratic program. □

Like the class of POPs treated in [11], the POP (8) satisfies some structured sparsity. With respect to the framework developed in [18] the sparsity structure of (8) is characterized by the chordal correlative sparsity pattern (CSP) matrix, which describes which $\psi_{i,j}\,(1,\leq i,j\leq N)$ and $\omega_{k,l}\,(1\leq k,l\leq N)$ are dependent on each other in CF(R,N) (C.f. Figure 1).

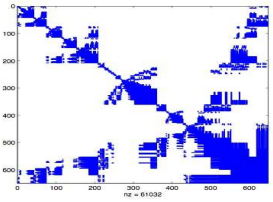


Figure 1: Chordal CSP matrix for $CF(R,20)$

Therefore we apply the methods proposed in [11] to solve CF(R,N), i.e. to approximate the solutions of (8) by solutions of a hierarchy of semidefinite program relaxations SDPR(w) [18], where w denotes the order of the semidefinite program (SDP) relaxation. In theory, the solution of SDPR(w) converges to the optimal solution for (8) for $w\to\infty$. Nevertheless, the capacity of present SDP solvers restricts the choice of the relaxation order w, as the size of SDPR(w) grows rapidly in w. However, as pointed out in [18] for many quadratic POPs it is sufficient to choose a relaxation order $w\in\{1,\ldots,4\}$ to approximate the POP's minimizer accurately.

REMARK 3. *It is a well known result (c.f. [10]), that SDPR(1) and (5) are equivalent, in the case POP (5) is a convex quadratic program. Thus, solving $CF(0,N)$ is equivalent to solving a SDP. Moreover, it is easy to show that the constraints admit only one feasible point when $R=0$.*

3.1 Improving the accuracy of SDPR(w)

As stated before, the solution of SDPR(w) converges to the optimizer of the POP for $w\to\infty$. Nevertheless, as the dimension n of CF(R,N) is given by $n=2(N-2)^2$, choosing a relaxation order w greater than 2 for a medium scale discretization N yields a SDP which requires too much memory in order to be solved by the used SDP-solver SeDuMi [17]. Therefore, we have to restrict ourselves to $w=1,2$ for small scale N, or even to $w=1$ for medium scale N. We cannot expect that SDPR(1) or SDPR(2) provide accurate approximations to the optimal solution for any R. In order to tighten the SDP relaxation, we impose lower and upper bounds lbd^ψ, lbd^ω, ubd^ψ and $\text{ubd}^\omega\in\mathbb{R}^{N^2}$ such that

$$\text{lbd}^\psi_i\leq\psi_i\leq\text{ubd}^\psi_i \text{ and } \text{lbd}^\omega_i\leq\omega_i\leq\text{ubd}^\omega_i \quad \forall\, 1\leq i\leq N^2. \tag{9}$$

And in order to improve the accuracy of the SDPR solution, we may apply additional locally convergent optimization techniques. For instance Newton's method for nonlinear systems can be applied to DSCF(R,N) where the SDPR(w) solution is taken as the starting point. Or alternatively, (8) is approximated by sequential quadratic programming (SQP) [2], again, with the SDPR(w) solution as starting point of the algorithm. Combining the sparse SDP relaxation with Newton's method or SQP is summarized in the scheme:

METHOD 1. **SDPR method**

1. *Choose the two parameters R and N.*
2. *Apply SDPR(w) to CF(R,N) and obtain solution $\tilde{u}:=(\tilde{\psi},\tilde{\omega})$.*
3. *Apply sequential quadratic programming (SQP) to CF(R,N) or Newton's method to DSCF(R,N), each of them starting from $\tilde{u}$, and obtain $u:=(\psi,\omega)$.*

3.2 Gröbner basis method and SDPR

The Gröbner basis method to find all complex solutions of a given system of zero dimensional polynomial equations is a useful tool for tuning the parameters of the SDPR and for validating its numerical results. In order to do this, we will study DSCF(R,N) by the rational univariate representation [13], [12], which is a variation of the Gröbner basis method, for coarse discretizations N. The 5×5 mesh ($N=5$) is solvable with this method (Groebner(Fgb) in Maple 11, nd_gr_trace and tolex_gsl in Risa/Asir). The system for the 5×5 mesh case contains 18 variables and 9 in the 18 appear as linear and the other 9 as quadratic variables. Applying Gröbner basis method to solve DSCF$(R,5)$ for different settings of R, and enumerating all solutions by their kinetic energy allows us to confirm whether the SDPR solutions are indeed the minimal energy solutions of DSCF(R,N) and to determine which relaxation order w is sufficient to derive this global minimal energy solution. The result is also used to tune parameters ε^k_i which appears in the next section. We have no theorem which states that the tuning based on the 5×5 case is good for the $n\times n$ case. However, we believe

this tuning provides a better approximation for the $n \times n$ case, too. Note, whereas the Gröbner basis method finds all complex solutions of DSCF(R,N), the SDPR method finds the real solution of DSCF(R,N) that minimizes F.

3.3 Enumeration algorithm for finding the k smallest energy solutions

As mentioned in Section 2, we conjecture the number of solutions of the discrete steady cavity flow problem DSCF(R,N) is finite, i.e. the feasible set of CF(R,N) is finite. Method 1 enables us to approximate the global minimal solution $u^\star = u^{(1)\star} := (\psi^{(1)\star}, \omega^{(1)\star})$ of CF(R,N). Beside the minimal solution, we are also interested in finding the solution $u^{(2)\star}$ with the second smallest kinetic energy, the solution $u^{(3)\star}$ with the third smallest kinetic energy or in general the solution $u^{(k)\star}$ with the kth smallest kinetic energy. Based on the SDPR method we propose an algorithm that enumerates the k smallest kinetic energy solutions of CF(R,N). Our algorithm shares the idea of separating the feasible set by additional constraints with Branch-and-Bound and cutting plane methods that are used for solving mixed integer linear programs and general concave optimization problems [7]. In contrast to the linear constraints of those methods we impose quadratic constraints to separate the feasible set. Moreover, CF(R,N) is a non-convex continuous quadratic optimization problem for $R \neq 0$. It may be worth investigating in future how extensions of Branch-and-Cut methods for certain nonconvex problems [5] can be used in our setting.

ALGORITHM 1. *Find the k smallest solutions approximately:*
Given $u^{(k-1)}$, the approximation to the $(k-1)$th energy solution obtained by solving SDPR$^{k-1}(w)$.

1. *Choose ϵ_1^k and $\epsilon_2^k > 0$.*
2. *Choose integers $b_1^k,\ b_2^k \in \{1,\ldots,(N-2)^2\}$.*
3. *Add the following quadratic constraints to SDPR$^{k-1}(w)$ and denote the resulting (tighter) SDP relaxation as SDPR$^k(w)$.*

$$\begin{aligned} (u_j - u_j^{(k-1)})^2 &\geq \epsilon_1^k \qquad \forall 1 \leq j \leq b_1^k, \\ (u_{j+(N-2)^2} - u_{j+(N-2)^2}^{(k-1)})^2 &\geq \epsilon_2^k \quad \forall 1 \leq j \leq b_2^k. \end{aligned} \tag{10}$$

4. *Solve SDPR$^k(w)$ with $w = 1, 2$ or larger, if possible. Obtain a first approximation $u^{SDP(k)}$.*
5. *Apply a local optimization technique as for instance Newton's method or SQP with $u^{SDP(k)}$ as starting point. Obtain $u^{(k)}$ as an approximation to $u^{(k)\star}$.*
6. *Iterate steps 1–5.*

Note that $u_j = \psi_j$ and $u_{j+(N-2)^2} = \omega_j$ if $1 \leq j \leq N$. The idea of Algorithm 1 is to impose an additional polynomial inequality constraint (10) to the POP (8) in iteration k, that excludes the previous iteration's solution $u^{(k-1)}$ from the feasible set of POP (8). In the case the feasible set of (8) is finite and $u^{(k-1)}$ is sufficiently close to $u^{(k-1)\star}$, the new constraint excludes $u^{(k-1)\star}$ from the feasible set of (8) and $u^{(k)\star}$ is the new global minimizer of (8). Of course, there are various alternatives to step 3 in Algorithm 1, in order to exclude $u^{(k-1)\star}$ from the POP's feasible set. One alternative constraint is

$$\left(u_i - u_i^{(k-1)\star}\right) u_{n+i} - \epsilon_i = 0 \quad (1 \leq i \leq b), \tag{11}$$

where $b \in \{1,\ldots,n\}$, $\epsilon_i > 0$ and u_{n+i} an additional slack variable bounded by -1 and 1. It is easy to see that (11) is violated, if $u = u^{(k-1)\star}$. Nevertheless, it turned out that the numerical performance of (11) is inferior to the one of (10) for our problem DSCF(R,N) as the tuning parameters ϵ_i and b is far more difficult for (11) compared to (10). A second alternative to exclude $u^{(k-1)\star}$ are l_p-norm constraints such as

$$\| u - u^{(k-1)\star} \|_p = \left(\sum_{i=1}^{n} \left(u_i - u_i^{(k-1)\star}\right)^p \right)^{\frac{1}{p}} \geq \epsilon, \tag{12}$$

for $p \geq 1$. The disadvantage of the constraints (12) is, they destroy the sparsity of the POP (8), as all u_i $(i = 1,\ldots,n)$ occur in the same constraint. Therefore the advantage of the sparse SDP relaxations is lost and the POP can not be solved efficiently anymore. These observations justify to impose (10) as additional constraints in Algorithm 1. We obtain the following results for Algorithm 1.

PROPOSITION 2. *Let R and N be fixed, $(u^{(1)},\ldots,u^{(k-1)})$ be the output of the first $(k-1)$ iterations of Algorithm (1). If this output is a sufficiently close approximation of the vector of $(k-1)$ smallest kinetic energy solutions $(u^{(1)\star},\ldots,u^{(k-1)\star})$, and if the feasible set of POP (8) is finite and distinct in terms of the energy function, i.e. $F(u^{(1)\star}) < F(u^{(2)\star}) < \ldots$, then there exist $b \in \{1,\ldots,n\}$ and $\epsilon \in \mathbb{R}^b$ such that the output $u^{(k)}$ of Algorithm 1 (for kth iteration) satisfies*

$$u^{(k)}(w) \to u^{(k)\star} \quad \text{when } w \to \infty. \tag{13}$$

Proof: .

As each $u^{(j)}$ is in a neighborhood of $u^{(j)\star}$ for all $j \in \{1,\ldots,k-1\}$, we can choose $b \in \{1,\ldots,n\}$ and a vector $\epsilon \in \mathbb{R}^b$, s.t.

$$\forall j \in \{1,\ldots,k-1\}\ \exists i \leq b \text{ s.t. } \left(u_i^{(j)} - u_i^{(j)\star}\right)^2 < \epsilon_i,$$

and for each $j \in \{1,\ldots,k-1\}$ holds

$$\left(u_i^{(j)} - u_i^{(l)\star}\right)^2 \geq \epsilon_i\ \forall l \geq k\ \forall i \in \{1,\ldots,b\}.$$

Let CF$(R,N)^{(k)}$ denote the CF(R,N) with the k systems of additional constraints given by step 3 in Algorithm 1, where the kth constraints are given by (10) for the constructed b and ϵ. Then it holds

$$\text{feas}\left(CF(R,N)^{(k)}\right) = \text{feas}\left(CF(R,N)\right) \backslash \left\{u^{(1)\star},\ldots,u^{(k-1)\star}\right\}.$$

Thus, $u^{(k)\star}$ is the global minimizer of $CF(R,N)^{(k)}$ and the global minimum is $F(u^{(k)\star})$. As the bounds (9) guarantee the compactness of the feasible set, it holds with the convergence theorem for the sparse SDP relaxations [9], if $w \to \infty$,

$$\begin{aligned} \min \text{SDPR}^{(k)}(w) &\to \min CF(R,N)^{(k)} = F(u^{(k)\star}), \\ u^{(k)}(w) &\to u^{(k)\star}. \qquad \square \end{aligned} \tag{14}$$

Although we have proven convergence, the capacity of current SDP solvers restricts the choice of the relaxation order w to small integers, in our application typically to $w = 1$ or $w = 2$. Furthermore, we need to choose the parameters ϵ and b appropriately, to obtain good approximations of the k smallest kinetic energy solutions. In the following numerical experiments, we see the Gröbner basis method is an useful tool to tune the two parameters ϵ and b, as it allows to confirm whether we derive the k smallest energy solutions successfully in case N is small ($N = 5$).

3.4 Numerical results

We will demonstrate the numerical performance of the SDPR(w) and Algorithm 1 to approximately enumerate the k smallest solutions. All calculations are conducted on a Mac OS X with CPU 2.5GHz and 2 GB Memory. As an implementation of the sparse SDP relaxation we use the software SparsePOP [15] and MATLAB optimization toolbox for standard SQP routines in order to improve the accuracy of the solution provided by SDPR(w). The total accumulated computation time in seconds is denoted by t_C, the scaled feasibility error of a SDPR solution u' w.r.t. the constraints of CF(R, N) by $\epsilon_{\rm sc}$.

3.4.1 CF(4000,5)

In a first setting we choose the discretization $N = 5$, i.e. the dimension of the POP (8) is $n = 2 \cdot 3^2 = 18$. This dimension is small enough to apply the Gröbner basis method to determine all complex solutions of DSCF(R, N). Therefore, we are able to verify whether the solutions provided by Algorithm 1 are optimal. The computational results are given in Table 1. Comparing our SDPR results to all solutions

k	w	ϵ_1^k	b_1^k	b_2^k	t_C	$\epsilon_{\rm sc}$	$F(u^{(k)})$	solution
0	1	-	-	-	2	2e-10	4.6e-4	$u^{(0)}$
1	1	1e-3	3	0	5	5e-4	6.3e-4	$u^{(1)}$
2	1	1e-3	3	0	8	5e-4	1.0e-3	$u^{(2)}$

Table 1: Results of Algorithm 1 for CF$(4000, 5)$

of the polynomial system obtained by polyhedral homotopy method or Gröbner basis method, it turns out that the solutions $u^{(0)}$, $u^{(1)}$ and $u^{(2)}$ indeed coincide with the three smallest energy solutions $u^{(0)\star}$, $u^{(1)\star}$ and $u^{(2)\star}$. The velocities (v_1, v_2) derived from these three solutions via (4) are displayed in Figure 2. Note, that the third smallest energy solution $u^{(2)}$ shows a vortex in counter-clockwise direction, which may indicate that this solution is a fake solution.

3.4.2 CF(20000,7)

We apply Algorithm 1 with SDPR(1) to CF$(20000, 7)$ and obtain the results in Table 2. The two parameter settings $(\epsilon_1^1, b_1^1) = (1e-3,\ 1)$ and $(\epsilon_1^1, b_1^1) = (1e-6,\ 5)$ are not sufficient to obtain an other solution than $u^{(0)}$, whereas $(\epsilon_1^1, b_1^1) = (1e-5,\ 5)$ yields $u^{(1)}$, a solution of larger energy. After another iteration with $(\epsilon_1^2, b_1^2) = (1e-5,\ 5)$ we obtain a third solution $u^{(3)}$ of even larger energy.

It is interesting to observe in Figure 3 that $u^{(1)}$ and $u^{(2)}$ are one-vortex solutions, whereas there seems to be no vortex in the smallest energy solution $u^{(0)}$.

3.4.3 CF(40000,7)

k	w	ϵ_1^k	b_1^k	b_2^k	t_C	$\epsilon_{\rm sc}$	$F(u^{(k)})$	solution
0	1	-	-	-	2	3e-7	3.4e-4	$u^{(0)}$
1	1	1e-3	1	0	5	5e-4	3.4e-4	$u^{(0)}$
1	1	1e-6	5	0	5	6e-6	3.4e-4	$u^{(0)}$
1	1	1e-5	5	0	9	5e-6	5.9e-4	$u^{(1)}$
2	1	1e-5	5	0	14	5e-6	5.2e-3	$u^{(2)}$

Table 2: Results of Algorithm 1 for CF$(20000, 7)$

Next, we examine $CF(40000, 7)$, which is a good example to demonstrate that solving DSCF(R, N) and POP (8) is becoming more difficult for larger Reynolds numbers. As for the previous problem, the dimension of the POP is $n = 50$, which is too large to be solved by Gröbner basis. Our computational results are reported in Table 3.

k	w	ϵ_1^k	b_1^k	b_2^k	t_C	$\epsilon_{\rm sc}$	$F(u^{(k)})$	solution
0	1	-	-	-	3	2e-7	3.4e-4	$u^{(0)}(1)$
1	1	5e-6	5	0	7	6e-9	7.3e-4	$u^{(1)}(1)$
2	1	5e-6	5	0	11	3e-6	5.9e-4	$u^{(2)}(1)$
3	1	8e-6	5	0	16	5e-6	2.3e-4	$u^{(3)}(1)$
0	2	-	-	-	5872	8e-10	2.6e-4	$u^{(0)}(2)$

Table 3: Results of Algorithm 1 for $CF(40000, 7)$

Solution $u^{(2)}(1)$ is of smaller energy than $u^{(1)}(1)$, and $u^{(3)}(1)$ is even of smaller energy than $u^{(0)}(1)$. This phenomenon can be explained by the fact, that the SDP relaxation with $w = 1$ is not tight enough to yield a solution that converges to $u^\star$ under the local optimization procedure. The energy of $u^{(0)}(2)$ obtained by SDPR(2) is smaller than the one of $u^{(0)}(1)$, but it is not the global minimizer as well. In fact, Algorithm 1 with SDPR(1) generates a better solution $u^{(3)}(1)$ (with smaller energy) in 3 iterations requiring 16 seconds computation time, compared to solution $u^{(0)}(2)$ obtained by applying SDPR(2) to $CF(40000, 7)$ requiring 5872 seconds. Thus, despite failing to enumerate the smalllest energy solutions in the right order with $w = 1$, applying our enumeration algorithm with relaxation order $w = 1$ is far more efficient than the original sparse SDP relaxation with $w = 2$ for approximating the global minimizer of POP (8). It is a future problem to make this construction systematic.

4. RELATIONS OF REYNOLDS NUMBERS R AND CF(R, N)

4.1 SDPR(w) for increasing discretization N

In our previous experiments we derived small or even minimal energy solutions by Method 1 and Algorithm 1 for various choices of the problem parameter R with discretization $N \in \{5,\ 6,\ 7\}$. In the case we succeed, applying SDPR(w) to CF(R, N) yields the minimal kinetic energy solution $u^\star$ of the discrete steady cavity flow problem. The important question arises whether it is possible to expand these coarse grid minimal kinetic energy solutions $u^\star$ to finer grids with larger discretization N, i.e. whether these coarse grid solutions converge to analytic solutions of the original (continu-

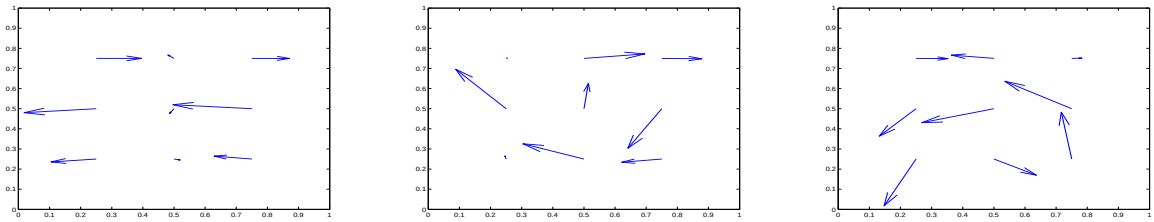


Figure 2: (v_1, v_2) **for solutions** $u^{(0)}$ **(left),** $u^{(1)}$ **(center) and** $u^{(2)}$ **(right) of CF**$(4000, 5)$ **on the interior of** $[0, 1]^2$

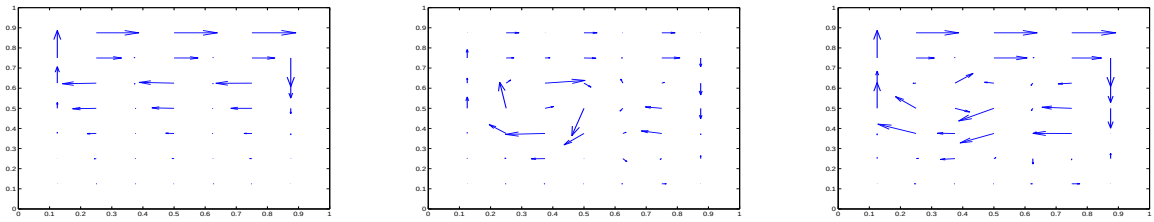


Figure 3: (v_1, v_2) **for solutions** $u^{(0)}$ **(left),** $u^{(1)}$ **(center) and** $u^{(2)}$ **(right) of CF**$(20000, 7)$ **on** $[0, 1]^2$

ous) steady cavity flow problem for $N \to \infty$. As pointed out in, e.g., [4], in the case of larger and larger Reynolds number R the discrete steady cavity flow problem has to be solved for finer and finer grids, in order to obtain solutions converging to solutions of the continuous problem for $N \to \infty$. As in section 4, the calculations are conducted on a Mac OS X with CPU 2.5GHz and 2 GB Memory.

4.1.1 *Small Reynolds number* R

We apply SDPR(1) to CF$(100, N)$ and take the solution as starting point for Newton's method. Accurate solutions with $\epsilon_{\text{sc}} < 1e-10$ to the discrete steady cavity flow problem are obtained for $N \in \{10, 15, 20\}$. By applying standard grid-refinement methods as in [11], we succeed in extending the solutions to grids of size 30×30 and 40×40, as pictured for $N = 40$ in Figure 4. Thus, it seems reasonable to conclude, that the minimal energy solution converges to an analytical solution of the steady cavity flow problem. The discrete steady cavity flow problem has multiple solutions. It is an advantage of our method to show that the minimal kinetic energy solution $u^\star$ converges to an analytic solution for $N \to \infty$.

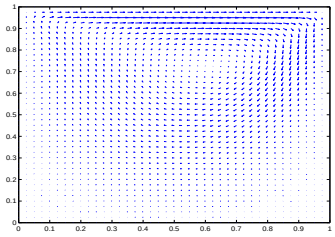


Figure 4: (v_1, v_2) **for solution** u **of CF**$(100, 40)$

4.1.2 *Large Reynolds number* R

Now, we examine $CF(10000, N)$ for $N \in \{8, \ldots, 18\}$. For all tested discretizations we were able to find accurate solutions by Method 1 with SDPR(1) and SQP.

If we compare the pictures in Figure 5, it seems the SDPR(1) solution of CF$(10000, N)$ evolves into some **stream-like** solution. Nevertheless, unlike the solutions of CF$(100, N)$, we have not been able to expand this solution to a grid of higher resolution by standard interpolation and grid-refinement methods so far. It is possible the solution pictured in Figure 5 is a fake solution.

QUESTION 1. *Does the minimal kinetic energy solution* $u^\star$ *of CF*(R, N) *converge to an analytic solution of the steady cavity flow problem for* $N \to \infty$*, even for large values of* R*?*

4.2 Solutions of CF(R, N) for increasing R

In order to address Question 1 and to understand why convergence to the analytic solution is a lot more difficult to obtain for large R, we examine the behavior of the minimal energy solution of the polynomial system DSCF(R, N) and CF(R, N), respectively, for increasing Reynolds number R.

4.2.1 *Minimal kinetic energy solution for increasing* R

Method 1 is one possible approach to solve DSCF(R, N). If w is chosen sufficiently large, the output u of Method 1 is guaranteed to accurately approximate the minimal energy solution $u^\star$ of CF(R', N) and DSCF(R', N), respectively. In order to show the advantage of the SDPR method we compare our results to solutions of DSCF(R', N) obtained by the following standard procedure:

METHOD 2. *Naive homotopy-like* **continuation method**

1. *Choose the parameters* R'*,* N *and a step size* ΔR*.*
2. *Solve DSCF*$(0, N)$*, i.e. a linear system, and obtain its unique solution* u^0*.*

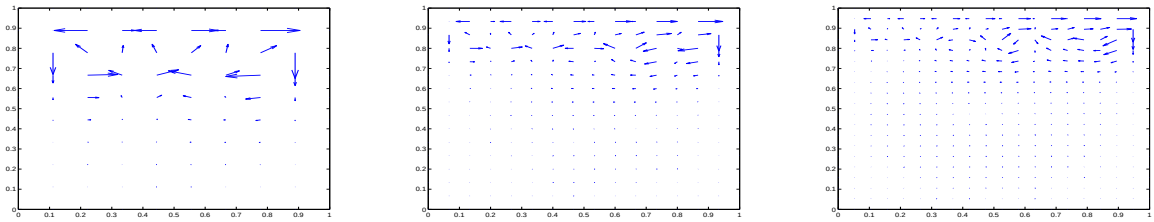


Figure 5: SDPR(1) solutions of $CF(10000, N)$ for $N = 8$ (left), $N = 14$ (center) and $N = 18$ (right)

3. *Increase R^{k-1} by ΔR: $R^k = R^{k-1} + \Delta R$*
4. *Apply Newton's method to DSCF(R^k, N) starting from u^{k-1}. Obtain solution u^k as an approximation to a solution of the discrete cavity flow problem.*
5. *Iterate 3. and 4. until the desired Reynold's number R' is reached.*

Note, the continuation method does not necessarily yield the minimal kinetic energy solution of DSCF(R, N). Let $u^\star(R, N)$ denote the global minimizer of CF(R, N), the minimal energy is given by $E_{\min}(R, N) = F(u^\star(R, N))$. Obviously, it holds $E_{\min}(0, N) = F(u_0(N))$.

In a next step the solution of DSCF(R, N) is obtained by the continuation method starting from u_0 is denoted as $\tilde{u}(R)$, and its energy as $E_C(R, N) := F(\tilde{u}(R, N))$. As illustrated for $N = 5$ in Figure 6, it is possible for all R to find a continuation $\tilde{u}$ of u_0. For $N = 5$ the dimension n of the discrete steady cavity flow problem is $n = 18$. This dimension is small enough to solve a polynomial system by Gröbner basis method and to determine all complex solutions of the system. Therefore, we can verify whether Method 1 detects the global minimizer of CF(R, N) or not. It is worth pointing out, that we are able to find the minimal energy solution of the CF(R, N) by applying the SDP relaxation method, whereas this solution cannot be obtained by the continuation method. We observe SDPR(1) is sufficient to detect the global optimizer for $R \leq 10000$, and for $R \geq 20000$ the global optimizer is obtained by SDPR(2), i.e. $E_{\min}(R, 5) = E_{\text{SDPR(2)}}(R, 5)$ for any R.

R	$N_{\mathbb{C}}$	$N_{\mathbb{R}}$	E_C	$E_{SDPR(1)}$	$E_{SDPR(2)}$
0	1	1	0.0096	0.0096	0.0096
100	37	13	0.0030	0.0030	0.0030
500	37	13	6.2e-4	6.2e-4	6.2e-4
1000	37	13	5.4e-4	5e-4	5e-4
2000	37	13	6.2e-4	6.2e-4	6.2e-4
4000	37	17	6.3e-4	4.6e-4	4.6e-4
6000	36	16	5.7e-4	4.5e-4	4.5e-4
8000	36	16	5.2e-4	4.5e-4	4.5e-4
10000	35	17	4.7e-4	4.5e-4	4.5e-4
30000	35	17	4.5e-4	4.5e-4	2.5e-4
100000	34	16	4.5e-4	4.5e-4	8.8e-5

Table 4: Numerical results for $CF(R, 5)$

In the case of $N = 6$ and $N = 7$ the dimension of the polynomial system is too large to be solved by Gröbner basis method for $R > 0$. For $N = 6$ the continuation method, SDPR(1) and SDPR(2) yield the same solution for all tested R. And in case of $N = 7$ the continuation solution $\tilde{u}(R)$ is detected by SDPR(1) as well, except the case $R = 6000$, where a solution with slightly smaller energy is detected by SDPR(1), as documented in Table 5.

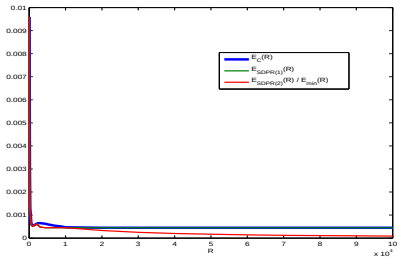


Figure 6: $E_C(R)$, $E_{\text{SDPR}(1)}(R)$, $E_{\text{SDPR}(2)}(R)$ and $E_{\min}(R)$ for $N = 5$

R	0	100	4000	6000	10000
E_C	2.0e-2	7.7e-3	4.1e-4	3.7e-4	3.4e-4
$E_{\text{SDPR}(1)}$	2.0e-2	7.7e-3	4.1e-4	3.6e-4	3.4e-4

Table 5: Numerical results for $CF(R, 7)$

Summarizing these results, $F(u_0(N)) \geq F(\tilde{u}(R, N))$ for any of the tested $R > 0$. It is an advantage of our approach to show, $\tilde{u}(R, N)$ is in general not the optimizer of $CF(R, N)$ for increasing R. In fact, for some settings we obtain far better approximations to the minimal energy solution than $\tilde{u}(R, N)$. Furthermore, $E_{min}(R)$ and $E_C(R)$ are both decreasing in R. The behavior of E_C, E_{SDPR} and $E_{\min}$ coincides for all chosen discretization N and motivates the following conjecture.

CONJECTURE 1. *Let discretization N be fixed.*

a) $F(u_0(N)) = E_{\min}(0, N) \geq E_{\min}(R, N) \geq 0 \quad \forall R \geq 0.$

b) $E_{\min}(R, N) \to 0$ *for* $R \to \infty$.

As an application, Conjecture 1 can be used as a certificate for the non-optimality of a feasible solution u' of CF(R, N) in case $F(u'(R, N)) > E_{\min}(0, N)$. If it is possible to extend u_0 to R via continuation method, $\tilde{u}(R, N)$ can serve as a non-optimality certificate in case $F(u'(R, N)) > F(\tilde{u}(R, N))$.

5. CONCLUSION

We proposed an algorithm to approximately enumerate all real solutions of a zero dimensional radical polynomial system with respect to a cost function, which takes advantage of the sparse semidefinite program relaxation method (SDPR) in order to find a good starting point for Newton's method or sequential quadratic programming. The algorithm can be applied successfully to the discrete cavity flow problem with the kinetic energy of the flow as cost function. We can guarantee the convergence of the algorithm's output to the smallest kinetic energy solutions of the polynomial system, if the order of the SDPR tends to infinity. Our numerical experiments for various choices of R have demonstrated that it is sufficient to apply SDPR of order one or two, in order to succeed in obtaining accurate approximations to the smallest energy solutions of the discrete cavity flow problem by our enumeration algorithm. In the case of small Reynolds numbers our algorithm allowed for another interesting observation: Among all solutions of the polynomial system given by the discrete cavity flow problem, the minimal kinetic energy solution converges to an analytic solution of the continuous steady cavity flow problem. In the case of large Reynolds number R we are not able to extend our coarse grid solutions to a finer grid, yet, although many of them look like stream solutions when the kinetic energy is small. It is known that the set of solutions of the discrete cavity flow problem contains lots of non-physical solutions or fake solutions, but there has been no systematic study of the discrete cavity flow problem as a polynomial system so far. Moreover, the more interesting stream-like solutions of the discrete steady cavity flow problem are usually among the 3rd or 4th smallest kinetic energy solutions. Our enumeration algorithm based on the SDPR method provides a powerful tool to detect the smallest energy solutions one by one. Further analysis of the polynomial system derived from the steady cavity flow problem for large Reynolds number R will remain an interesting topic in future.

To conclude, we think that the polynomial system of the discrete steady cavity flow problem is challenging for the community of solvers of polynomial systems and numerical algebra. Another interesting challenge is to solve the discrete steady cavity flow problem derived by the alternative finite difference discretization of the Jacobian proposed by Arakawa [1]. For its observed and described properties the discrete steady cavity flow problem will be a good test problem to validate new techniques for solving systems of algebraic equations and inequalities. Furthermore, as solving the cavity flow problem for large Reynolds numbers R remains an active field of research, we believe that our numerical results may be instructive for audiences in the community of numerical analysis for fluid dynamics to understand fake solutions in partial differential equations.

6. ACKNOWLEDGMENTS

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