MIR Closures of Polyhedral Sets

Sanjeeb Dash*
Oktay Günlük†
Andrea Lodi‡

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Abstract

We study the mixed-integer rounding (MIR) closures of polyhedral sets. The MIR closure of a polyhedral set is equal to its split closure and the associated separation problem is NP-hard. We describe a mixed-integer programming (MIP) model with linear constraints and a non-linear objective for separating an arbitrary point from the MIR closure of a given mixed-integer set. We linearize the objective using additional variables to produce a linear MIP model that solves the separation problem exactly. Using a subset of these additional variables yields an MIP model which solves the separation problem approximately, with an accuracy that depends on the number of additional variables used. Our analysis yields an alternative proof of the result of Cook, Kannan and Schrijver (1990) that the split closure of a polyhedral set is again a polyhedron. We also discuss a heuristic to obtain MIR cuts based on our approximate separation model, and present some computational results.

1 Introduction

In this paper we study the mixed-integer rounding (MIR) closure of a given mixed-integer set

$$P = \{v \in R^{|J|},\ x \in Z^{|I|}\ :\ Cv + Ax \geq b,\ v, x \geq 0\}$$

where all numerical data is rational. In other words, we are interested in the set of non-negative points that satisfy all MIR inequalities

$$(\lambda C)^+ v + (-\lambda)^+ (Cv + Ax - b) + \min\{\lambda A - \lfloor \lambda A \rfloor, r\mathbf{1}\}x + r\lfloor \lambda A \rfloor x \geq r \lceil \lambda b \rceil$$

that can be generated by some λ of appropriate dimension. Here $r = \lambda b - \lfloor \lambda b \rfloor$, $(\cdot)^+$ denotes $\max\{0, \cdot\}$, $\mathbf{1}$ is an all-ones vector, and all operators are applied to vectors component-wise.

*IBM T.J. Watson Research Center, P.O. Box 218, Yorktown Heights, NY 10598 sanjeebd@us.ibm.com
†IBM T.J. Watson Research Center, P.O. Box 218, Yorktown Heights, NY 10598 gunluk@us.ibm.com
‡DEIS, University of Bologna, viale Risorgimento 2, 40136 Bologna, Italy Supported in part by the EU projects ADONET (contract n. MRTN-CT-2003-504438) and ARRIVAL (contract n. FP6-021235-2). andrea.lodi@unibo.it

In Section 2, we discuss in detail how these inequalities are derived and why they are called MIR inequalities.

The term *mixed-integer rounding* was first used by Nemhauser and Wolsey [26, pp.244] to denote valid inequalities that can be produced by what they call the MIR procedure. These authors in [25] strengthen and redefine the MIR procedure and the resulting inequality. The same term was later used to denote seemingly simpler inequalities in Marchand and Wolsey [24], and Wolsey [28]. In this paper we give a comprehensive review of the different definitions of MIR inequalities and clarify the relationship between them. The definition of the MIR inequality we use in this paper is equivalent to the one in [25], though our presentation is based on [28].

Split cuts were defined by Cook, Kannan and Schrijver in [15], and are a special case of the disjunctive cuts introduced by Balas [3]. In [25], Nemhauser and Wolsey show that MIR cuts are equivalent to split cuts in the sense that, for a given polyhedral set described by linear equations, every MIR cut is a split cut and vice-versa. In this paper, we show that this does not hold for inequality systems unless slack variables are explicitly taken into account. This has also been independently observed by Bonami and Cornuéjols [10] recently. In [15], Cook, Kannan and Schrijver show that the split closure (the set of points satisfying all split cuts) of a polyhedral set is again a polyhedron. Alternative proofs of this result were given by Andersen, Cornuéjols and Li [1], and more recently by Vielma [27]. In this paper, we present an alternative – in our view significantly simpler – short proof of the same fact by analyzing MIR closures of polyhedral sets.

Caprara and Letchford [13] show that separating an arbitrary point from the split closure of a polyhedral set is NP-hard. A similar property was shown by Eisenbrand [21] for the Chvátal closure of a polyhedral set. Bonami and Minoux [12] approximately optimize over the rank-1 lift-and-project closure of 0-1 mixed integer programs; in this setting, the separation problem can be framed as a linear program. Independently, Fischetti and Lodi [23] show that, in practice, it is possible to separate points from the Chvátal closure in a reasonable amount of time. Their approach involves formulating the separation problem as an MIP, and solving it with a general MIP solver. By repeatedly applying their separation algorithm, they are able to approximately optimize over the Chvátal closures of MIPLIB instances and obtain very tight bounds on the value of optimal solutions. Motivated by the above work, and the fact that the MIR closure is contained (usually strictly) in the Chvátal closure or lift-and-project closure (for 0-1 problems), we describe an MIP model for separating from the MIR closure of a polyhedral set exactly. Our exact MIP model is unlikely to be a practical tool because of its size; we also describe an MIP model (by dropping some of the variables in the previous model) for approximate separation. We present computational results on approximately optimizing over the MIR closure for problems in the MIPLIB 3.0 test set. Our computational work is different in spirit from that of Fischetti and Lodi [23]; we use our approximate MIR separation model in conjunction with other heuristics to find violated MIR cuts. Our work is related to the paper of Balas and Saxena [7] (written independently) who solve MIPs to obtain violated split cuts and approximately optimize over the split closure of a polyhedral set. They obtain strong bounds on the optimal values of many MIPLIB 3.0 instances in this manner. In Section 3.3 we discuss their model in detail.

The paper is organized as follows. In Section 2, we define MIR inequalities and discuss how our definition is related to earlier definitions. In Section 3, we present a non-linear integer program for separating MIR inequalities. We establish the equivalance of this model with the (non-linear) separation models for split cuts presented by Caprara and Letchford [13] and Balas and Saxena [7]. We also present a linear mixed-integer programming model that approximately separates an arbitrary point from the MIR closure of a given polyhedral set. In Section 4, we present a simple proof that the MIR (or, split) closure of a polyhedral set is again a polyhedron. Further, we present an MIP model for exact MIR separation. In Sections 5 and 6 we discuss computational issues and present a summary of our computational experiments with a heuristic which combines our approximate separation model with other MIR separation heuristics.

2 Mixed-integer rounding inequalities

In this section we discuss MIR inequalities and define the MIR closure of a polyhedral set. We also present a basic result that shows that the MIR closure is invariant under simple variable transformations.

2.1 The Basic Mixed-Integer Inequality

In [28], Wolsey develops the MIR inequality as the only non-trivial facet of the following simple mixed-integer set:

$$Q^0 = \Big\{ v \in R,\ x \in Z \ :\ v + x \ \geq \ b, \ \ v \geq 0 \Big\}$$

where $b \notin Z$. It is easy to see that

$$v \geq \hat{b}(\lceil b \rceil - x), \tag{1}$$

where $\hat{b} = b - \lfloor b \rfloor$ is the fractional part of b, is valid and facet defining for Q^0. In [28], this inequality is called the *basic mixed-integer* inequality.

To apply this idea to more general sets defined by a single inequality, one needs to combine variables to get a structure resembling Q^0. More precisely, given a set

$$Q^1 = \Big\{ v \in R^{|J|},\ x \in Z^{|I|} \ :\ \sum_{j \in J} c_j v_j + \sum_{i \in I} a_i x_i \ \geq \ b, \ \ v,\ x \geq 0 \Big\}$$

the defining inequality is relaxed to obtain

$$\Big(\sum_{j \in J} \max\{0, c_j\} v_j + \sum_{i \in I'} \hat{a}_i x_i\Big) + \Big(\sum_{i \in I \setminus I'} x_i + \sum_{i \in I} \lfloor a_i \rfloor x_i\Big) \geq \ b$$

where $\hat{a}_i = a_i - \lfloor a_i \rfloor$ and $I' \subseteq I$. As the first part of the left hand side of this inequality is non-negative, and the second part is integral, the MIR inequality

$$\sum_{j \in J} \max\{0, c_j\} v_j + \sum_{i \in I'} \hat{a}_i x_i \geq \ \hat{b}\Big(\lceil b \rceil - \sum_{i \in I \setminus I'} x_i - \sum_{i \in I} \lfloor a_i \rfloor x_i\Big)$$

is valid for Q^1. Notice that $I' = \{i \in I \ : \ \hat{a}_i < \hat{b}\}$ gives the strongest inequality of this form and therefore the MIR inequality can also be written as

$$\sum_{j\in J}(c_j)^+v_j + \sum_{i\in I}\min\{\hat{a}_i, \hat{b}\}x_i + \hat{b}\sum_{i\in I}\lfloor a_i\rfloor x_i \geq \ \hat{b}\lceil b\rceil \tag{2}$$

where $(\cdot)^+$ denotes $\max\{0, \cdot\}$ as defined earlier.

2.2 Aggregating constraints

For sets defined by $m > 1$ inequalities, one can combine the m inequalities to obtain a single *base* inequality and then apply inequality (2) to the base inequality. Let

$$P = \Big\{v \in R^l, \ x \in Z^n \ : \ Cv + Ax \geq b, \ v, x \geq 0\Big\}$$

be a mixed-integer set where C, A and b are vectors of appropriate dimension. To obtain the base inequality, one possibility is to use a vector $\lambda \in R^m$, $\lambda \geq 0$ to combine the inequalities defining P. This approach leads to the base inequality

$$\lambda Cv + \lambda Ax \ \geq \ \lambda b$$

and the corresponding MIR inequality

$$(\lambda C)^+v + \min\{\lambda A - \lfloor\lambda A\rfloor, r\mathbf{1}\}x + r\lfloor\lambda A\rfloor x \geq r\lceil\lambda b\rceil, \tag{3}$$

where operators $(\cdot)^+$, $\lfloor\cdot\rfloor$ and $\min\{\cdot,\cdot\}$ are applied to vectors component-wise, and $r = \lambda b - \lfloor\lambda b\rfloor$.

Alternatively, one can first convert the inequalities defining P into equations by introducing slack variables, and then combine the equations using a vector λ which is not necessarily non-negative. This leads to the base inequality

$$\lambda Cv + \lambda Ax - \lambda s \ = \ \lambda b$$

and the corresponding MIR inequality

$$(\lambda C)^+v + (-\lambda)^+s + \min\{\lambda A - \lfloor\lambda A\rfloor, r\mathbf{1}\}x + r\lfloor\lambda A\rfloor x \geq r\lceil\lambda b\rceil, \tag{4}$$

where s denotes the (non-negative) slack variables. Finally, substituting out the slack variables gives the following MIR inequality in the original space of P:

$$(\lambda C)^+v + (-\lambda)^+(Cv + Ax - b) + \min\{\lambda A - \lfloor\lambda A\rfloor, r\mathbf{1}\}x + r\lfloor\lambda A\rfloor x \geq r\lceil\lambda b\rceil. \tag{5}$$

These inequalities are what we call MIR inequalities in this paper.

Notice that when $\lambda \geq 0$, inequality (5) reduces to inequality (3). When $\lambda \ngeq 0$, however, there are inequalities (5) which cannot be written in the form (3). We present an example to emphasize this point.

Example 1 *Consider the following simple mixed-integer set*

$$T = \{v \in R,\ x \in Z\ :\ -v - 4x \geq -4,\quad -v + 4x \geq 0,\quad v, x \geq 0\}$$

and the base inequality generated by $\lambda = [-1/8, 1/8]$

$$x + s_1/8 - s_2/8 \geq 1/2$$

where s_1 *and* s_2 *denote the slack variables for the first and second constraint, respectively. The corresponding MIR inequality is* $1/2x + s_1/8 \geq 1/2$*, which after substituting out* s_1*, becomes* $-v/8 \geq 0$ *or simply* $v \leq 0$*. This inequality defines the only non-trivial facet of* T*.*

It is not possible to generate this inequality if slacks are not used, and (thereby) the multipliers are restricted to be non-negative. A base inequality generated by $\lambda_1, \lambda_2 \geq 0$ *has the form*

$$(-\lambda_1 - \lambda_2)v + (-4\lambda_1 + 4\lambda_2)x \geq -4\lambda_1.$$

with v *having a non-positive coefficient. Therefore, the MIR inequality (3) generated by this base inequality would have a coefficient of zero for* v*, establishing that* $v \leq 0$ *cannot be generated as an MIR.*

We note that a similar example is also independently developed in [10]. Also see Cornuéjols[16] for a discussion of various valid inequalities for integer programs including MIR inequalities.

2.3 Basic properties of MIR inequalities

Let P^{LP} denote the continuous relaxation of P. A linear inequality $hv + gx \geq d$ is called a *split cut* for P if it is valid for both $P^{LP} \cap \{\bar{\alpha}x \leq \bar{\beta}\}$ and $P^{LP} \cap \{\bar{\alpha}x \geq \bar{\beta} + 1\}$, where $\bar{\alpha}$ and $\bar{\beta}$ are integral. The inequality $hv + gx \geq d$ is said to be derived from the *disjunction* $\bar{\alpha}x \leq \bar{\beta}$ and $\bar{\alpha}x \geq \bar{\beta} + 1$. Obviously all points in P satisfy any split cut for P. Note that multiple split cuts can be derived from the same disjunction.

The basic MIR inequality (1) is a split cut for Q^0 derived from the disjunction $x \leq \lfloor b \rfloor$ and $x \geq \lfloor b \rfloor + 1$. Therefore, the MIR inequality (5) is also a split cut for P derived from the disjunction $\bar{\alpha}x \leq \bar{\beta}$ and $\bar{\alpha}x \geq \bar{\beta} + 1$ where $\bar{\beta} = \lfloor \lambda b \rfloor$ and

$$\bar{\alpha}_i = \begin{cases} \lceil (\lambda A)_i \rceil & \text{if } (\lambda A)_i - \lfloor (\lambda A)_i \rfloor \geq \lambda b - \lfloor \lambda b \rfloor \\ \lfloor (\lambda A)_i \rfloor & \text{otherwise.} \end{cases}$$

This observation also implies that if a point $(v^*, x^*) \in P^{LP}$ violates the MIR inequality (5) then $\bar{\beta} + 1 > \bar{\alpha}x^* > \bar{\beta}$.

Nemhauser and Wolsey [25] showed that every split cut for P can be derived as an MIR cut for P. As we show later, what we call MIR inequalities in this paper are equivalent to the MIR inequalities defined in [25]. We next formally define the MIR closure of a polyhedral set.

Definition 2 *The MIR closure of* P *is the set of points in* P^{LP} *which satisfy all MIR inequalities (5) that can be generated by some multiplier vector* $\lambda \in R^m$*.*

Thus, the split closure of a polyhedral set is the same as its MIR closure.

2.4 Original MIR procedure of Nemhauser and Wolsey

In their book [26, Section II.1.6], Nemhauser and Wolsey develop the MIR inequalities for mixed-integer sets. Both the inequalities that define these sets and the MIR inequalities derived for them are given in the "$\leq$" form. To compare their inequality with what we call the MIR inequality in this paper, we present their results in the "$\geq$" form.

Let $P = \{v \in R^l,\ x \in Z^n\ :\ Cv + Ax \geq b,\ v, x \geq 0\}$ as before. The MIR procedure of Nemhauser and Wolsey starts with two vectors $\lambda^1, \lambda^2 \geq 0$ of appropriate dimension to generate two implied inequalities

$$\lambda^1 Cv + \lambda^1 Ax \geq \lambda^1 b \ \text{ and } \ \lambda^2 Cv + \lambda^2 Ax \geq \lambda^2 b.$$

Using these two *base* inequalities, the procedure then generates the following valid MIR inequality:

$$\lambda^1 Ax + r\lceil \lambda^2 A - \lambda^1 A \rceil x + \max\{\lambda^1 C, \lambda^2 C\} v \geq r\lceil \lambda^2 b - \lambda^1 b \rceil + \lambda^1 b$$

where $r = \lambda^2 b - \lambda^1 b - \lfloor \lambda^2 b - \lambda^1 b \rfloor$. This inequality can also be written as follows:

$$\big((\lambda^2 - \lambda^1)C\big)^+ v + \lambda^1(Cv + Ax - b) + r\lceil(\lambda^2 - \lambda^1)A\rceil x \geq r\lceil(\lambda^2 - \lambda^1)b\rceil. \qquad (6)$$

Notice that, given a vector λ and the associated MIR inequality (5), it is possible to construct two non-negative vectors $\lambda^2 = (\lambda)^+$ and $\lambda^1 = (-\lambda)^+$ and produce the corresponding inequality (6). The two inequalities would look identical, except some of the coefficients of the integer variables would be stronger in inequality (5) due to the term $\min\{\lambda A - \lfloor \lambda A \rfloor, r\mathbf{1}\}x$. Similarly, given two vectors $\lambda^1, \lambda^2 \geq 0$, it is possible to show that MIR inequality (5) generated by $\lambda = \lambda^2 - \lambda^1$ dominates inequality (6).

2.5 Revised MIR procedure of Nemhauser and Wolsey

Later in [25], Nemhauser and Wolsey extend their earlier result to produce valid inequalities for $P' = \{v \in R^{|J|},\ x \in Z^{|I|}\ :\ C'v + A'x \geq b'\}$ where the variables are not explicitly required to be non-negative. More precisely, they show that given two multiplier vectors $\mu^1, \mu^2 \geq 0$ that satisfy (i) $\mu^1 C' = \mu^2 C'$ and $(ii)(\mu^2 - \mu^1)A' \in Z$, the MIR inequality (6) generated by these vectors is valid for P'. In this case, inequality (6) becomes

$$\mu^1(C'v + A'x - b') + r'(\mu^2 - \mu^1)A'x \geq r'\lceil(\mu^2 - \mu^1)b'\rceil \qquad (7)$$

where $r' = (\mu^2 - \mu^1)b' - \lfloor(\mu^2 - \mu^1)b'\rfloor$. Notice that if both μ^1_i and μ^2_i are strictly positive for some index i, inequality (7) can be strengthened by decreasing both multipliers. It is therefore possible to let $\mu = \mu^2 - \mu^1$ and write (a strengthening of) inequality (7) as

$$(-\mu)^+(C'v + A'x - b') + r'\mu A'x \geq r'\lceil \mu b' \rceil \qquad (8)$$

where the vector μ is not restricted in sign and it satisfies (i) $\mu C' = 0$ and (ii) $\mu A'$ is integral.

We next show that inequality (8) and the MIR inequality (5) are equivalent when applied to the set P in the sense that for any λ it is possible to construct an appropriate

μ that would give the same inequality and vice-versa. Notice that the non-negativity requirements are not explicitly present in the definition of P'. It is possible to represent the set P in this form by defining

$$C' = \begin{bmatrix} C \\ I \\ 0 \end{bmatrix}, \quad A' = \begin{bmatrix} A \\ \mathbf{0} \\ I \end{bmatrix}, \quad b' = \begin{bmatrix} b \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

where I and $\mathbf{0}$ respectively denote the identity and zero matrix of appropriate dimension.

Let λ be given and consider $\mu = [\lambda, -\lambda C, \gamma]$ where

$$\gamma_i = \begin{cases} -\hat{a}_i & \text{if } \hat{a}_i < r \\ 1 - \hat{a}_i & \text{otherwise} \end{cases},$$

and $\hat{a} = \lambda A - \lfloor \lambda A \rfloor$. Note that $\mu C' = 0$ and $\mu A'$ is integral. Also notice that $\mu b' = \lambda b$ and therefore $r' = r$. Inequality (8) for this choice of μ is

$$(-\lambda)^+(Cv + Ax - b) + (\lambda C)^+ v + (-\gamma)^+ x + r(\lambda A + \gamma)x \geq r\lceil \lambda b \rceil$$

where the coefficient of x can also be written as

$$(-\gamma)^+ + r\lfloor \lambda A \rfloor + r(\hat{a} + \gamma) \;=\; r\lfloor \lambda A \rfloor + \min\{\hat{a}, r\mathbf{1}\}.$$

Therefore, inequality (8) generated by μ is identical to inequality (5) generated by λ.

Conversely, given $\mu = [\mu_0, \mu_v, \mu_x] \geq 0$ consider the corresponding inequality (8)

$$(-\mu_0)^+(Cv + Ax - b) + (-\mu_v)^+ v + (-\mu_x)^+ x + r'(\mu_0 Ax + \mu_x x) \geq r'\lceil \mu_0 b \rceil$$

and notice that $\mu C' = 0$ implies that $\mu_0 C = -\mu_v$ and therefore $(-\mu_v)^+ = (\mu_0 C)^+$. In addition, $r' = \mu_0 b - \lfloor \mu_0 b \rfloor$. As $\mu A'$ is integral, $(\mu_0 A + \mu_x)$ is integral and therefore $\mu_x = -\hat{a} + t$ where t is an integral vector. Clearly inequality (8) can be strengthened unless $t_i = 0$ if $\hat{a}_i < r$ and $t_i = 1$, otherwise. It is therefore clear that the MIR inequality (5) generated by μ_0 is identical to inequality (8) generated by μ.

We next give a basic property of MIR inequalities (8) for the set P'. This property is known to hold for the Chvátal closure [21] and can easily be extended to MIR cuts.

Proposition 3 *The MIR closure of P' is invariant under the operation $y = Ux + l$ where l is an integer vector and U is a unimodular matrix.*

Proof Let $clo(\cdot)$ denote the MIR closure of a set. We will show that a given point $(\bar{v}, \bar{x}) \in clo(P')$ if and only if $(\bar{v}, \bar{y}) \in clo(T)$ where $\bar{y} = U\bar{x} + l$ and $T = \{v \in R^{|J|},\ y \in Z^{|I|} \,:\, C'v + A'U^{-1}y \geq b' + A'U^{-1}l\}$.

Assume that $(\bar{v}, \bar{x}) \in clo(P')$ and $(\bar{v}, \bar{y}) \notin clo(T)$. Then there exists a μ such that

$$(-\mu)^+(C'\bar{v} + A'U^{-1}\bar{y} - b' - A'U^{-1}l) + r(\mu A'U^{-1})\bar{y} < r\left\lceil \mu(b' + A'U^{-1}l) \right\rceil$$

where r denotes the fractional part of $\mu(b' + A'U^{-1}l)$, and $\mu C' = 0$ and $\mu A'U^{-1}$ is integral. This implies that $\mu A'U^{-1}l$ is integral and therefore r is also equal to the fractional part of $\mu b'$. As $\bar{y} = U\bar{x} + l$, the above inequality can also be written as

$$(-\mu)^+(C'\bar{v} + A'\bar{x} - b') + r(\mu A')\bar{x} + r\mu A'U^{-1}l < r\left\lceil \mu b' + \mu A'U^{-1}l \right\rceil.$$

Furthermore, as $\mu A'U^{-1}l$ is integral, $(-\mu)^+(C'\bar{v} + A'\bar{x} - b') + r(\mu A')\bar{x} < r\lceil \mu b' \rceil$. This, however, cannot be true as $\bar{x}$ must satisfy the MIR inequality generated by the same μ.

Similarly, it is possible to show that $\bar{x} \notin clo(P')$ and $\bar{y} \in clo(T)$ leads to a contradiction. ∎

3 The Separation Problem

In this section, we study the problem of separating an arbitrary point from the MIR closure of the polyhedral set $P = \{v \in R^l,\ x \in Z^n\ :\ Cv + Ax \geq b,\ v, x \geq 0\}$. In other words, for a given point, we are interested in either finding violated inequalities or concluding that none exists. For convenience of notation, we first argue that without loss of generality we can assume P is given in equality form.

Consider the MIR inequality (4) for P,

$$(\lambda C)^+ v + (-\lambda)^+ s + \min\{\lambda A - \lfloor \lambda A \rfloor, r\mathbf{1}\}x + r\lfloor \lambda A \rfloor x \geq r\lceil \lambda b \rceil,$$

where s denotes the slack expression $(Cv + Ax - b)$. If we explicitly define the slack variables, by letting $\tilde{C} = (C,\ -I)$ and $\tilde{v} = (v,\ s)$, then the constraints defining P become $\tilde{C}\tilde{v} + Ax = b,\ \tilde{v} \geq 0,\ x \geq 0$, and the MIR inequality can be written as

$$(\lambda \tilde{C})^+ \tilde{v} + \min\{\lambda A - \lfloor \lambda A \rfloor, r\mathbf{1}\}x + r\lfloor \lambda A \rfloor x \geq r\lceil \lambda b \rceil. \tag{9}$$

In other words, all continuous variables, whether slack or structural, can be treated uniformly. In the remainder of this paper we assume that P is given in the equality form

$$P = \{v \in R^l, x \in Z^n : Cv + Ax = b, v, x \geq 0\}.$$

We denote the continuous relaxation of P by P^{LP}.

3.1 Relaxed MIR inequalities

Let

$$\begin{aligned}
\Pi \;=\; \Big\{(\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in R^m \times R^l \times R^n \times Z^n \times R \times Z \quad &: \\
c^+ \;&\geq\; \lambda C \\
\hat{\alpha} + \bar{\alpha} \;&\geq\; \lambda A \\
\hat{\beta} + \bar{\beta} \;&\leq\; \lambda b \\
c^+ \;&\geq\; 0 \\
1 \;\geq\; \hat{\alpha} \;&\geq\; 0 \\
1 \;\geq\; \hat{\beta} \;&\geq\; 0\Big\}.
\end{aligned}$$

Note that for any $(\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$,

$$c^+ v + (\hat{\alpha} + \bar{\alpha})x \geq \hat{\beta} + \bar{\beta} \tag{10}$$

is valid for P^{LP} as it is a relaxation of $(\lambda C)v + (\lambda A)x = \lambda b$. Furthermore, using the basic mixed-integer inequality (1), we infer that

$$c^+ v + \hat{\alpha}x + \hat{\beta}\bar{\alpha}x \geq \hat{\beta}(\bar{\beta} + 1) \tag{11}$$

is a valid inequality for P. We call inequality (11) where $(\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$ a *relaxed MIR inequality* derived using the *base inequality* (10). We next show some basic properties of relaxed MIR inequalities.

Lemma 4 *A relaxed MIR inequality (11) violated by $(v^*, x^*) \in P^{LP}$ satisfies*

(*i*) $1 > \hat{\beta} > 0$,

(*ii*) $1 > \Delta > 0$,

(*iii*) *the violation of the inequality is at most* $\hat{\beta}(1 - \hat{\beta}) \leq 1/4$,

where $\Delta = \bar{\beta} + 1 - \bar{\alpha}x^*$ *and violation is defined to be the right hand side of inequality (11) minus its left hand side.*

Proof If $\hat{\beta} = 0$, then the relaxed MIR cut is trivially satisfied by all points in P^{LP}. Furthermore, if $\hat{\beta} = 1$, then inequality (11) is identical to its base inequality (10) which again is satisfied by all points in P^{LP}. Therefore, a non-trivial relaxed MIR cut satisfies $1 > \hat{\beta} > 0$.

For part (ii) of the Lemma, note that if $\bar{\alpha}x^* \geq \bar{\beta} + 1$ then inequality (11) is satisfied, as $c^+, \hat{\alpha}, \hat{\beta} \geq 0$ and $(v^*, x^*) \geq 0$. Furthermore, if (v^*, x^*) satisfies inequality (10) and $\bar{\alpha}x^* \leq \bar{\beta}$, then so is inequality (11) as $\hat{\beta} \leq 1$. Therefore, as the cut is violated, $1 > \Delta > 0$. It is also possible to show this by observing that inequality (11) is a split cut for P derived from the disjunction $\Delta \geq 1$ and $\Delta \leq 0$.

For the last part, let $w = c^+ v^* + \hat{\alpha}x^*$ so that the base inequality (10) becomes $w \geq \hat{\beta} + \Delta - 1$ and the relaxed MIR inequality (11) becomes $w \geq \hat{\beta}\Delta$. Clearly

$$\hat{\beta}\Delta - w \leq \hat{\beta}(w + 1 - \hat{\beta}) - w = \hat{\beta}(1 - \hat{\beta}) - (1 - \hat{\beta})w \leq \hat{\beta}(1 - \hat{\beta}).$$

The last inequality follows from the fact that $w \geq 0$ and $\hat{\beta} \leq 1$. ∎

Next, we relate MIR inequalities to relaxed MIR inequalities.

Lemma 5 *For any $\lambda \in R^m$, the MIR inequality (9) is a relaxed MIR inequality.*

Proof For a given multiplier vector λ, define α to denote λA. Further, set $c^+ = (\lambda C)^+$, $\bar{\beta} = \lceil \lambda b \rceil$ and $\hat{\beta} = \lambda b - \lfloor \lambda b \rfloor$. Also, define $\hat{\alpha}$ and $\bar{\alpha}$ as follows:

$$\hat{\alpha}_i = \begin{cases} \alpha_i - \lfloor \alpha_i \rfloor & \text{if } \alpha_i - \lfloor \alpha_i \rfloor < \hat{\beta} \\ 0 & \text{otherwise} \end{cases}, \qquad \bar{\alpha}_i = \begin{cases} \lfloor \alpha_i \rfloor & \text{if } \alpha_i - \lfloor \alpha_i \rfloor < \hat{\beta} \\ \lceil \alpha_i \rceil & \text{otherwise} \end{cases},$$

Clearly, $(\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$ and the corresponding relaxed MIR inequality (11) is the same as the MIR inequality (9). ∎

Lemma 6 *MIR inequalities dominate relaxed MIR inequalities.*

Proof Let $(v^*, x^*) \in P^{LP}$ violate a relaxed MIR inequality $\mathcal{I}$ generated with $(\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$. We will show that (v^*, x^*) also violates the MIR inequality (9).

Due to Lemma 4, we have $\bar{\beta} + 1 - \bar{\alpha}x^* > 0$ and therefore increasing $\hat{\beta}$ only increases the violation of the relaxed MIR inequality. Assuming $\mathcal{I}$ is the most violated relaxed MIR inequality, $\hat{\beta} = \min\{\lambda b - \bar{\beta}, 1\}$. By Lemma 4, we know that $\hat{\beta} < 1$, and therefore $\hat{\beta} = \lambda b - \bar{\beta}$ and $\bar{\beta} = \lfloor \lambda b \rfloor$.

In addition, due to the definition of Π we have $c^+ \geq (\lambda C)^+$ and $\hat{\alpha} + \hat{\beta}\bar{\alpha} \geq \min\{\lambda A - \lfloor \lambda A \rfloor, \hat{\beta}\mathbf{1}\} + \hat{\beta} \lfloor \lambda A \rfloor$. As $(v^*, x^*) \geq 0$, the violation of the MIR inequality is at least as much as the violation of $\mathcal{I}$. ∎

Combining Lemmas 5 and 6, we observe that a point in P^{LP} satisfies all MIR inequalities, if and only if it satisfies all relaxed MIR inequalities. In other words we have shown the following:

Corollary 7 *The MIR closure of P is the set of points in P^{LP} which satisfy all relaxed MIR inequalities (11) that can be generated by some $(\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$.*

Therefore, it is posible to define the MIR closure of a polyhedral set without using operators that take minimums, maximums or extract fractional parts of numbers. Let $\bar{\Pi}$ be the projection of Π in the space of $c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}$ and $\bar{\beta}$ variables. In other words, $\bar{\Pi}$ is obtained by projecting out the λ variables. We now describe the MIR closure of P as follows:

$$P^{MIR} = \Big\{(v,x) \in P^{LP} \ : \ c^+v + \hat{\alpha}x + \hat{\beta}\bar{\alpha}x \geq \hat{\beta}(\bar{\beta}+1) \text{ for all } (c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \bar{\Pi}\Big\}.$$

We would like to emphasize that $\bar{\Pi}$ is not the polar of P^{MIR} and therefore even though $\bar{\Pi}$ is a polyhedral set (with a finite number of extreme points and extreme directions), we have not yet shown that the polar of P^{MIR} is polyhedral. The polar of a polyhedral set is defined to be the set of points that yield valid inequalities for the original set. If the original set is defined in R^n, its polar is defined in R^{n+1} and the first n coordinates of any point in the polar give the coefficients of a valid inequality for the original set, and the last coordinate gives the right hand side of the valid inequality. Therefore, the polar of P^{MIR} is the collection of points $(c^+, \hat{\alpha} + \hat{\beta}\bar{\alpha}, \hat{\beta}(\bar{\beta}+1)) \in R^{l+n+1}$ where $(c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \bar{\Pi}$. A set is polyhedral if and only if its polar is polyhedral.

For a given point $(v^*, x^*) \in P^{LP}$, testing if $(v^*, x^*) \in P^{MIR}$ can be achieved by solving the following non-linear integer program (MIR-SEP):

$$\begin{array}{ll} \max & \hat{\beta}(\bar{\beta}+1) - (c^+v^* + \hat{\alpha}x^* + \hat{\beta}\bar{\alpha}x^*) \\ \text{s.t.} & \\ & (\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \ \in \ \Pi. \end{array}$$

If the optimal value of this program is non-positive, then $(v^*, x^*) \in P^{MIR}$. On the other hand, if the optimal value is positive, the optimal solution gives a most violated MIR inequality.

3.2 An Approximate separation model

We next (approximately) linearize the nonlinear terms that appear in the objective function of MIR-SEP. To this end, we first define a new variable Δ that stands for the term $(\bar{\beta} + 1 - \bar{\alpha}x)$. We then approximate $\hat{\beta}$ by a number $\tilde{\beta} \le \hat{\beta}$ representable over some $\mathcal{E} = \{\epsilon_k : k \in K\}$. We say that a number δ is *representable* over $\mathcal{E}$ if $\delta = \sum_{k \in \bar{K}} \epsilon_k$ for some $\bar{K} \subseteq K$. We can therefore write $\tilde{\beta}$ as $\sum_{k \in K} \epsilon_k \pi_k$ using binary variables π_k and approximate $\hat{\beta}\Delta$ by $\tilde{\beta}\Delta$ which can now be written as $\sum_{k \in K} \epsilon_k \pi_k \Delta$. Finally, we linearize terms $\pi_k \Delta$ using standard techniques as π_k is binary and $\Delta \in (0, 1)$ for any violated inequality.

An approximate MIP model Appx-MIR-Sep for the separation of the most violated MIR inequality reads as follows:

$$\max \quad \sum_{k \in K} \epsilon_k \Delta_k - (c^+ v^* + \hat{\alpha} x^*) \tag{12}$$

$$\text{s.t.} \quad (\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi \tag{13}$$

$$\hat{\beta} \ge \sum_{k \in K} \epsilon_k \pi_k \tag{14}$$

$$\Delta = (\bar{\beta} + 1) - \bar{\alpha} x^* \tag{15}$$

$$\Delta_k \le \Delta \qquad \forall k \in K \tag{16}$$

$$\Delta_k \le \pi_k \qquad \forall k \in K \tag{17}$$

$$\pi \in \{0, 1\}^{|K|} \tag{18}$$

Let z^{sep} and $z^{apx-sep}$ denote the optimal value of MIR-SEP and Appx-MIR-Sep, respectively. For any integral solution of Appx-MIR-Sep, we have $(\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$ and

$$\sum_{k \in K} \epsilon_k \Delta_k \le \sum_{k \in K} \epsilon_k \Delta \pi_k$$

establishing that $z^{sep} \ge z^{apx-sep}$. In other words, Appx-MIR-Sep is a restriction of MIR-SEP and if the approximate separation problem finds a solution with objective function value $z^{apx-sep} > 0$, the corresponding MIR cut is violated by at least as much.

In our computational experiments, we use $\mathcal{E} = \{2^{-k} : k = 1, \ldots, \bar{k}\}$ for some small number $\bar{k}$. We next show that with this choice of $\mathcal{E}$, Appx-MIR-Sep yields a violated MIR cut provided that there is an MIR cut with a "large enough" violation. Notice that for any $\hat{\beta}$ there exists a $\tilde{\beta}$ representable over $\mathcal{E}$ such that $2^{-\bar{k}} \ge \hat{\beta} - \tilde{\beta} \ge 0$.

Theorem 8 *Let $\mathcal{E} = \{2^{-k} : k = 1, \dots, \bar{k}\}$ for some positive integer $\bar{k}$, then*

$$z^{sep} \;\geq\; z^{apx-sep} \;>\; z^{sep} - 2^{-\bar{k}} \tag{19}$$

where z^{sep} and $z^{apx-sep}$ denote the optimal values of MIR-SEP and Appx-MIR-Sep, respectively.

Proof The first inequality holds as Appx-MIR-Sep is a restriction of MIR-SEP. For the second inequality, note that $z^{apx-sep} \geq 0$ as we can get a feasible solution of Appx-MIR-Sep with objective 0 by setting Δ to 1, and the remaining variables to 0. Therefore the second inequality in (19) holds if $z^{sep} \leq 0$. Assume that $z^{sep} > 0$. Let $(\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$ be an optimal solution of MIR-SEP. For the variables in Appx-MIR-Sep common with MIR-SEP, set their values to the above optimal solution of MIR-SEP. Let $\tilde{\beta}$ be the largest number representable over $\mathcal{E}$ less than or equal to $\hat{\beta}$. Clearly, $2^{-\bar{k}} \geq \hat{\beta} - \tilde{\beta} \geq 0$. Choose $\pi \in \{0,1\}^{\bar{k}}$ such that $\tilde{\beta} = \sum_{k \in K} \epsilon_k \pi_k$. Set $\Delta = \bar{\beta} + 1 - \bar{\alpha}x^*$. Set $\Delta_k = 0$ if $\pi_k = 0$, and $\Delta_k = \Delta$ if $\pi_k = 1$. Then $\Delta_k = \pi_k \Delta$ for all $k \in K$, and $\tilde{\beta}\Delta = \sum_{k \in K} \epsilon_k \Delta_k$. Therefore,

$$2^{-\bar{k}} \;>\; 2^{-\bar{k}}\Delta \;\geq\; \hat{\beta}\Delta - \tilde{\beta}\Delta = \hat{\beta}\Delta - \sum_{k \in K} \epsilon_k \Delta_k.$$

The second inequality in (19) follows. ∎

In the next section (Theorem 15) we show that Appx-MIR-Sep becomes an exact model for finding violated MIR cuts when $\mathcal{E}$ is chosen as $\{\epsilon_k = 2^k/\Phi, \forall k = \{1, \dots, \lceil log\Phi \rceil\}\}$ where Φ is the least common multiple of all subdeterminants of $A|C|b$.

3.3 Other separation models

Caprara and Letchford [13], and, more recently, Balas and Saxena [7] presented optimization models for finding a violated split cut for P. In both papers, the authors use two sets of multipliers that guarantee that the split cut is valid for both sides of the disjunction; see equations (8)-(13) in [13] and system (SP) in [7]. Caprara and Letchford (resp. Balas and Saxena) denote the split cut by $\alpha x + \beta y \leq \gamma$ (resp. $\alpha x \geq \beta$) and the corresponding disjunction by $cx \leq d$ and $cx \geq d + 1$ (resp. $\pi x \leq \pi_0$ and $\pi x \geq \pi_0 + 1$). In addition, both papers use a "normalization" condition restricting the sum of the multipliers for the inequalities in the disjunction to be a constant. In the case of Balas and Saxena, the sum of the multipliers u_0 and v_0 for the inequalities $\pi x \leq \pi_0$ and $\pi x \geq \pi_0 + 1$, respectively, is restricted to be 1, whereas the corresponding sum in [13] is restricted to be 2.

It is possible to show that the separation models in the above papers – equations (8)-(13) in [13], and system (2.1) or (PMILP) in [7] – actually find the MIR cut (7) that has the largest violation (left hand side minus right hand side). To see this for the model in [13], let $[A, G]$ in [13] stand for $[-A', -C']$, and $b, \mu^L, \mu^R, \lambda^R$ in [13] stand for $-b', 2\mu^2, 2\mu^1, 2r'$, respectively. With these transformations, the objective function (equation (8) in [13]) of the Caprara-Letchford model equals 4*(left hand side - right hand side of (7)).

Similarly, for the Balas-Saxena model, let A in [7] stand for $[A', C']$, and b, u, v, u_0 in [7] stand for $b', \mu^2, \mu^1, 1 - r'$, respectively. Then the objective function in (PMILP) is simply the left hand side of (7) minus its right hand side. Therefore, we have the following result.

Lemma 9 *The following three models have the same set of optimal solutions: (i) the Caprara-Letchford model given by equations (8)-(13) in [13], (ii) the Balas-Saxena model given by system (2.1) or (PMILP) in [7], and, (iii) MIR-SEP.*

It is interesting to note that the normalization used in [13] and [7] is implicitly built into the definition of the MIR cut.

Caprara and Letchford do not perform any computational tests with their model. As for Balas and Saxena, instead of linearizing the product $\hat{\beta}\Delta$ as we do, they fix the term u_0 (corresponding to $1 - \hat{\beta}$) in their model to a small set of values from [0,0.5], and solve an MIP for each value. Their linearization approach is very similar to ours except our model imposes a lower bound on $\hat{\beta}$ from a small set of values. To highlight this difference, consider the following example where $P = \{v \in R,\ x \in Z\ :\ v + x \geq 0.31,\ v \geq 0\}$ and the point to be separated is $(v^*, x^*) = (0, 0.31)$. Clearly, the convex hull is given by adding the simple MIR cut $v + 0.31x \geq 0.31$ which is violated by (v^*, x^*), with a violation of $0.31(1 - 0.31)$. Using our linearized separation model with $k = 2$, i.e., $\epsilon_1 = 0.5, \epsilon_2 = 0.25$, there exists a solution to our model with $\lambda = 1$ that gives the simple MIR cut above. For this solution, the objective value of the model is $0.25 * (1 - 0.31)$ which is an underestimate of the cut violation. (Using $k > 2$ gives a better aproximation.) The Balas/Saxena model PMILP (or, system (2.1)) for this example (or more precisely, the deparametrized model MILP(θ)) is infeasible unless the parameter θ (or, u_0) is chosen to be exactly 0.31.

One other difference between the Balas-Saxena model and ours is that in MIR-SEP we use only one set of multipliers corresponding to the inequalities defining P.

4 A simple proof that the MIR closure is a polyhedron

In this section we give a short proof that the MIR closure of a polyhedral set is a polyhedron. As MIR cuts are equivalent to split cuts, this result obviously follows from the work of Cook, Kannan and Schrijver (1990) on split cuts. Andersen, Cornuéjols and Li (2005), and Vielma (2006) give alternative proofs that the split closure of a polyhedral set is a polyhedron. We believe our proof is simpler than the previous proofs; further it is framed in the language of MIR cuts and not split cuts.

The main tool in the proof is a finite bound on the multipliers λ needed for non-redundant MIR cuts given in Lemma 12. The bounds on λ can be tightened if the MIP is a pure integer program, and we give these tighter bounds first, in the next lemma. In this section we assume that the coefficients in $Cv + Ax = b$ are integers. Denote the ith equation of $Cv + Ax = b$ by $c_i v + a_i x = b_i$. An equation $c_i v + a_i x = b_i$ is a *pure integer* equation if $c_i = 0$.

Lemma 10 *If some MIR inequality is violated by the point (v^*, x^*), then there is another MIR inequality violated by (v^*, x^*) derived using $\lambda_i \in [0, 1)$ for every pure integer equation.*

Proof: (sketch) Let $(\lambda, (\lambda C)^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$ define an MIR inequality where $\lambda_i \notin [0, 1)$ for a pure integer equation $c_i v + a_i x = b_i$ where $c_i = 0$. It is possible to show that the MIR inequality defined by

$$(\lambda - \lfloor \lambda_i \rfloor e_i, (\lambda C)^+, \hat{\alpha}, \bar{\alpha} - \lfloor \lambda_i \rfloor a_i, \hat{\beta}, \bar{\beta} - \lfloor \lambda_i \rfloor b_i) \in \Pi$$

has precisely the same violation. ■

We note that it is possible to obtain a slightly weaker bound on the multipliers, (namely, $\lambda_i \in (-1, 1)$ for every pure integer equation) by combining Lemma 1 in [13] with the transformations described in Section 3.3.

Definition 11 *We define Ψ to be the largest absolute value of subdeterminants of C, and $1/m$ if $C = 0$, where m is the number of rows in $Ax + Cv = b$.*

Lemma 12 *If there is an MIR inequality violated by the point (v^*, x^*), then there is another MIR inequality violated by (v^*, x^*) with $\lambda_i \in (-m\Psi, m\Psi)$, where m is the number of rows in $Ax + Cv = b$.*

Proof: Let the MIR cut

$$(\lambda C)^+ v + \hat{\alpha} x + \hat{\beta} \bar{\alpha} x \geq \hat{\beta}(\bar{\beta} + 1) \tag{20}$$

be violated by (v^*, x^*). Then $(\lambda, (\lambda C)^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$ with $0 < \hat{\beta} < 1$. Let C_j stand for the jth column of C. Let $S_1 = \{j : \lambda C_j > 0\}$ and $S_2 = \{j : \lambda C_j \leq 0\}$.

Consider the following cone:

$$\mathcal{C} = \{v \in R^m : vC_i \leq 0 \;\; \forall i \in S_1, \;\; vC_i \geq 0 \;\; \forall i \in S_2\}.$$

Obviously λ belongs to $\mathcal{C}$. We will find a vector λ' in $\mathcal{C}$, such that $\bar{\lambda} = \lambda - \lambda'$ is integral and belongs to $\mathcal{C}$. $\mathcal{C}$ is a polyhedral cone, and is generated by a finite set of vectors $\mu_1, \ldots, \mu_t$, for some $t > 0$. (Observe that if $C = 0$, then $\mathcal{C} = R^m$, and $\mu_1, \ldots, \mu_t$ can be chosen to be the unit vectors times ± 1.) We can assume these vectors are integral (by scaling); we can also assume the coefficients of $\mu_1, \ldots, \mu_t$ have absolute value at most Ψ. Further, we can assume that $\mu_1, \ldots, \mu_k$ (here $k \leq m$) are linearly independent vectors such that

$$\lambda = \sum_{j=1}^{k} v_j \mu_j, \text{ with } v_j \in R, \; v_j > 0.$$

If $v_j < 1$ for $j = 1, \ldots, k$, then each coefficient of λ has absolute value less than $m\Psi$, and there is nothing to prove. If $v_j \geq 1$ for any $j \in \{1, \ldots, k\}$, then let

$$\lambda' = \sum_{j=1}^{k} \hat{v}_j \mu_j \Rightarrow \lambda - \lambda' = \sum_{j=1}^{k} \lfloor v_j \rfloor \mu_j,$$

where $\hat{v}_j = v_j - \lfloor v_j \rfloor$. Clearly λ' belongs to $\mathcal{C}$, and has coefficients with absolute value at most $m\Psi$. Also, $\lambda' \neq 0$ as $\lambda' = 0 \Rightarrow \lambda$ is integral $\Rightarrow \hat{\beta} = 0$. Let $\bar{\lambda} = \lambda - \lambda'$; obviously $\bar{\lambda}$ belongs to $\mathcal{C}$ and is integral. Further,

$$(\lambda C)^+ - (\lambda' C)^+ = (\bar{\lambda} C)^+.$$

Therefore $(\lambda', (\lambda' C)^+, \hat{\alpha}, \bar{\alpha} - \bar{\lambda} A, \hat{\beta}, \bar{\beta} - \bar{\lambda} b) \in \Pi$. It follows that the multipliers λ' lead to the MIR

$$(\lambda' C)^+ v + \hat{\alpha} x + \hat{\beta}(\bar{\alpha} - \bar{\lambda} A) x \geq \hat{\beta}(\bar{\beta} - \bar{\lambda} b + 1). \tag{21}$$

The rhs of the old MIR minus the rhs of the new MIR equals

$$\begin{aligned} \hat{\beta}\bar{\lambda} b = \hat{\beta}\bar{\lambda}(Ax^* + Cv^*) &= \hat{\beta}\bar{\lambda} Ax^* + \hat{\beta}\bar{\lambda} Cv^* \\ &\leq \hat{\beta}\bar{\lambda} Ax^* + \hat{\beta}(\bar{\lambda} C)^+ v^*. \end{aligned} \tag{22}$$

The lhs of the old MIR (with v^*, x^* substituted) minus the lhs of the new MIR equals the last term in (22). Therefore the new MIR is violated by at least as much as the old MIR and the lemma follows. ■

As the multipliers λ are bounded, there are only a finite number of choices for $\bar{\alpha}$ and $\bar{\beta}$ for non-redundant MIR cuts, see (23).

Theorem 13 *If there is an MIR inequality violated by the point (v^*, x^*), then there is another MIR inequality violated by (v^*, x^*) for which $\hat{\beta}$ and the components of $\lambda, \hat{\alpha}$ are rational numbers with denominator equal to a subdeterminant of $A|C|b$, and each component of λ is contained in the interval $[-m\Psi, m\Psi]$.*

Proof Let (v^*, x^*) be a point in P^{LP} which violates an MIR cut. Let this MIR cut be defined by $(\lambda_o, c_o^+, \hat{\alpha}_o, \bar{\alpha}_o, \hat{\beta}_o, \bar{\beta}_o) \in \Pi$. By Lemma 12 , we can assume each component of λ_o lies in the range $(-m\Psi, m\Psi)$. Define $\Delta_o = \bar{\beta}_o + 1 - \bar{\alpha}_o^T x^*$. Then

$$\hat{\beta}_o \Delta_o - c_o^+ v^* - \hat{\alpha}_o x^* > 0.$$

Consider the following LP:

$$\begin{aligned} \max \hat{\beta}\Delta_o - c^+ v^* &- \hat{\alpha} x^* \\ (\lambda, c^+, \hat{\alpha}, \bar{\alpha}_o, \hat{\beta}, \bar{\beta}_o) &\in \Pi \\ -m\Psi \leq \lambda_i &\leq m\Psi \end{aligned}$$

Note that the objective is a linear function as Δ_o is fixed. Further, we have fixed the variables $\bar{\alpha}$ and $\bar{\beta}$ in the constraints defining Π. The bounds on λ come from Lemma 12, except that we weaken them to non-strict inequalities. This LP has at least one solution for $(\lambda, c^+, \hat{\alpha}, \hat{\beta})$ with positive objective value, namely $(\lambda_o, c_o^+, \hat{\alpha}_o, \hat{\beta}_o)$. Therefore a basic optimal solution of this LP has positive objective value. Consider the MIR cut defined by an optimal solution along with $\bar{\alpha}_o$ and $\bar{\beta}_o$. It is obviously an MIR cut with violation at least the violation of the original MIR cut. Therefore, $0 < \hat{\beta} < 1$. Further, it is easy to see that the LP constraints (other than the bounds on the variables) can be written as

$$\begin{bmatrix} A^T & -I & & \\ C^T & & -I & \\ b^T & & & -1 \end{bmatrix} \begin{pmatrix} \lambda \\ \hat{\alpha} \\ c^+ \\ \hat{\beta} \end{pmatrix} \begin{matrix} \le \\ \le \\ \ge \end{matrix} \begin{pmatrix} \bar{\alpha}_o \\ 0 \\ \bar{\beta}_o \end{pmatrix}.$$

The theorem follows. ∎

By Theorem 13, each non-redundant MIR inequality is defined by multipliers $\lambda = (\lambda_i)$ where λ_i is a rational number between $-m\Psi$ and $m\Psi$ with a denominator equal to a subdeterminant of $A|C|b$. Therefore the number of non-redundant MIR inequalities is finite.

Corollary 14 *The MIR closure of a polyhedral set P is a polyhedron.*

As the MIR closure equals the split closure, it follows that the split closure of a polyhedral set is again a polyhedron. Let the split closure of P be denoted by $P_S = \bigcap_{c \in Z^n, d \in Z} P_{(c,d)}$, where for $c \in Z^n$ and $d \in Z$,

$$P_{(c,d)} = conv\{(P \cap \{cx \le d\}) \cup (P \cap \{cx \ge d+1\})\}.$$

Lemma 12 gives a characterization of the useful disjunctions in the definition of the split closure. Define the vector $\mu \in R^m$ by

$$\mu_i = \begin{cases} m\Psi & \text{if } c_i \neq 0 \\ 1 & \text{if } c_i = 0 \end{cases}$$

Define

$$D = \{(c,d) \in Z^n \times Z : -\mu|A| \le c \le \mu|A|, \lfloor -\mu|b| \rfloor \le d \le \lfloor \mu|b| \rfloor\}. \tag{23}$$

D is clearly a finite set, and

$$P_S = \bigcap_{c \in Z^n, d \in Z} P_{(c,d)} = \bigcap_{(c,d) \in D} P_{(c,d)}.$$

To see this, let x^* be a point in P but not in P_S. Then some split cut, which is also an MIR cut, is violated by x^*. By Lemma 12 , there is an MIR cut with $-\mu < \lambda < \mu$ which is violated by x^*. This MIR cut has the form $(\lambda C)^+ v + \hat{\alpha} x + \hat{\beta}\bar{\alpha} x \ge \hat{\beta}(\bar{\beta}+1)$, where $(\bar{\alpha}, \bar{\beta}) \in D$. Thus x^* does not belong to $P_{(\bar{\alpha},\bar{\beta})}$. This implies that

$$\bigcap_{(c,d) \in D} P_{(c,d)} \subseteq \bigcap_{c \in Z^n, d \in Z} P_{(c,d)},$$

and the two sets in the expression above are equal as the reverse inclusion is true by definition.

Theorem 15 *Let Φ be the least common multiple of all subdeterminants of $A|C|b$, $K = \{1, \dots, log\Phi\}$, and $\mathcal{E} = \{\epsilon_k = 2^k/\Phi, \forall k \in K\}$. Then Appx-MIR-Sep is an exact model for finding violated MIR cuts.*

Proof By Theorem 13, $\hat{\beta}$ in a violated MIR cut can be assumed to be a rational number with a denominator equal to a subdeterminant of $A|C|b$ and therefore of Φ. But such a $\hat{\beta}$ is representable over $\mathcal{E}$. ∎

5 Computational Issues

We next discus some practical issues that we encountered during our computational experiments.

5.1 Numerical Issues

Assume that the point (v^*, x^*) to be separated from the MIR closure of P is obtained by optimizing a linear function over P^{LP} using a practical LP solver. Then (v^*, x^*) will only be approximately feasible for P^{LP}, i.e., some of the inequalities defining P^{LP} will be violated by small amounts (usually at most 10^{-6}). MIR-SEP can then return cuts which are not useful. For example, if $v_i^* < 0$ for some index i, then the objective function of MIR-SEP, $\hat{\beta}\Delta - c^+v^* - \hat{\alpha}x^*$, can be made positive by setting λ to 0, and c_i^+ to a large positive number. Clearly, such a λ does not yield a violated MIR cut. Moreover, if some equation in $Cv + Ax = b$ is violated – let $c_iv + a_ix = b_i$ be the ith equation in $Cv + Ax = b$ and let $c_iv^* + a_ix^* < b_i$ – then MIR-SEP would choose a large positive value for λ_i. The resulting base inequality $c^+v + (\hat{\alpha} + \bar{\alpha})x \geq \hat{\beta} + \bar{\beta}$ would be violated by (v^*, x^*), and so would the associated MIR cut; the MIR cut would not necessarily be violated if we moved to another approximately feasible solution (v', x') of $Cv + Ax = b$ with $c_iv' + a_ix' \geq b_i$.

We deal with such issues by modifying (v^*, x^*) and b to get a truly feasible solution of a modified set of constraints. We let $v' = \max\{v^*, \mathbf{0}\}$, and $x' = \max\{x^*, \mathbf{0}\}$, for non-negative variables and then define b' as $Cv' + Ax'$. We use Appx-MIR-Sep to separate (v', x') from the MIR closure of $Cv + Ax = b', v, x \geq 0, x \in Z$. We use the multipliers λ in the solution of Appx-MIR-Sep to compute an MIR cut for P. In some cases this cut is not violated by (v^*, x^*), but this happens infrequently as (v', x') is usually close to (v^*, x^*).

5.2 Reducing the size of the separation problem

The number of integer variables in Appx-MIR-Sep equals the number of integer variables in P plus the number of variables π_i used in linearizing the objective; thus solving Appx-MIR-Sep could be as hard as solving the original MIP. However, violated MIR cuts can often be found by solving an MIP with fewer integer variables. Cook, Kannan and Schrijver [15] showed that the split closure of a face F of P equals the intersection of F with the split closure of P. Therefore, if (v^*, x^*) lies on a face F, then (v^*, x^*) violates a split cut for F, if and only if it violates a split cut for P. A specific approach to choosing F, and then obtaining a violated split cut for P is given in [4] and [5]. Given the point (v^*, x^*), they solve a separation problem in the space of variables which lie strictly between their bounds.

To see how the above approach works in our context, note that in Appx-MIR-Sep, the variables $c_i^+, \hat{a}_j, \bar{a}_j$ corresponding to $v_i^* = 0$ and $x_j^* = 0$ do not contribute to the objective. One can remove them and the corresponding constraints from Appx-MIR-Sep, solve the reduced Appx-MIR-Sep, and then compute their values from the multipliers λ in a solution to the reduced model. The resulting cut would have the same violation as the cut in the reduced set of variables. Further, if $x_j^* = 0$ for an index j, and P has an upper bound for x_j, say $u_j > 0$, then the component of λ corresponding to $x_j \leq u_j$ can be assumed to be 0. Finally, if $x_j^* = u_j$ and $x_j \leq u_j$ for points in P, we can replace x_j by $u_j - x_j'$ where $0 \leq x_j' \leq u_j$, derive an MIR cut for the modified system of constraints (here (v^*, x^*) maps to a point with $x_j' = 0$) and get an MIR cut for P by replacing x_j' by $u_j - x_j$.

For many problems in MIPLIB 3.0, Appx-MIR-Sep cannot be solved without adopting the above approach, e.g., **nw04**, which has 36 constraints and over 87000 0-1 variables. With this approach when $\bar{k} = 5$, the first separation MIP would have at most 36+5 integer variables, instead of 87000+5.

5.3 Finding good MIR cuts

Given a point $(v^*, x^*) \in P^{LP}$, the separation model MIR-SEP is guaranteed to produce the most violated MIR inequality, if there is one. Similarly, based on Theorem 8, the approximate model is guaranteed to produce an MIR inequality with violation slightly less than the most violated inequality. In both cases violation of a cut defined by $\kappa = (c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \bar{\Pi}$ is defined to be

$$\eta(\kappa) \;=\; \hat{\beta}\Delta - c^+ v^* - \hat{\alpha} x^*$$

where $\Delta = \bar{\beta} + 1 - \bar{\alpha} x^*$. Unfortunately, there is no guarantee that MIR cuts with maximum values of $\eta(\kappa)$ would actually be the most effective MIR cuts in practice.

Example 16 *Consider separating* $(x^*, y^*) = (0.001, 0.5)$ *from the MIR closure of*

$$P = \{x, y \in Z \;:\; 100x - y \geq -0.4,\; 100x + y \geq 0.6\}.$$

First we convert the inequalities defining P *to equations by adding slacks:*

$$\begin{aligned} 100x - y - s_1 &= -0.4, \quad (A) \\ 100x + y - s_2 &= 0.6, \quad (B) \end{aligned}$$

and construct the related point $(x^*, y^*, s_1^*, s_2^*) = (0.001, 0.5, 0, 0)$ *to be separated.*

The base inequality $s_1/2 + y \geq 1/2$ *can obtained by taking* $\lambda_A = -1/2$ *and* $\lambda_B = 1/2$. *The corresponding cut* $s_1/2 + y/2 \geq 1/2$ *has violation* 0.25. *This inequality can also be written as* $x \geq 0.006$ *after substituting out* s_1.

Another base inequality $x \geq 0.001$ *can be obtained by taking* $\lambda_A = \lambda_B = 1/200$. *The resulting MIR cut* $0.001x \geq 0.001$ *(or* $x \geq 1$*) has violation less than* 0.001. ∎

Another possible measure of violation for MIR inequalities is

$$\eta'(\kappa) \;=\; \Delta - \frac{c^+v^* + \hat{\alpha}x^*}{\hat{\beta}};$$

see [28]. In the previous example, η' of $x \geq 1$ is 0.999, whereas η' of $x \geq 0.006$ is only 0.006. This suggests that η' may be a more effective measure than η. However, consider the base inequality $s_1+x+2y \geq 1.001$ obtained by taking $\lambda_A = (-1+1/200)$ and $\lambda_B = (1+1/200)$. The resulting MIR cut $s_1 + 0.001(x+2y) \geq 0.002$ has a violation of 0.999. However, this inequality is even weaker than $x \geq 0.006$ after substituting out s_1.

Another problem with both these measures is that adding integral multiples of tight constraints without continuous variables to the original base inequality does not change the violation of the resulting MIR cut (see the proof of Lemma 10). For example, if $x^* = 0.5$, the base inequalities $x \geq .5$ and $11x \geq 5.5$ lead to MIR cuts with identical violation for each measure. The first inequality leads to $x \geq 1$ and the second one to $11x \geq 6$ which is clearly weaker than $x \geq 1$. It is possible to avoid this problem by normalizing the cut violation using the norm of the cut; however it is hard to incorporate this idea into a linear separation model.

6 Computational experiments

In this section we discuss our computational experiments with our approximate separation model. We start off with the continuous relaxation of a given problem instance and iteratively strengthen it with MIR cuts to (approximately) optimize over the MIR closure. For any fixed precision, it is possible to approximately optimize over the MIR closure using Appx-MIR-Sep. This, however, might not happen in a reasonable amount of time and therefore, our approach should be considered as a heuristic.

After some initial testing, we realized that using Appx-MIR-Sep alone to find violated MIR inequalities improves the lower bound very slowly. To speed up this process we implemented several heuristics to find solutions to Appx-MIR-Sep by focusing on certain sub-classes of MIR cuts. These solutions might be sub-optimal with respect to the objective function of Appx-MIR-Sep, but they help increase the performance of the algorithm significantly. As discussed in Section 5.3, the objective function used in Appx-MIR-Sep does not necessarily help produce the most effective cuts.

We next discuss some practical issues and describe the heuristic ideas that we used to speed up the algorithm. Finally, we present numerical results.

6.1 Modeling Issues

Practical MIPs, such as those in MIPLIB 3.0, do not necessarily have the same form as P. Many of the variables have upper bounds in addition to the lower bounds of 0. We simply treat the upper bound constraints as general linear constraints. Further, some of the variables have negative lower bounds. For an integer variable x_i bounded below by l_i, where l_i is a negative integer, we "shift" it by performing the substitution $x_i' = x_i - l_i$.

Finally, if an integer variable x_i is free, we replace the constraint $\hat{\alpha}_i + \bar{\alpha}_i \geq (\lambda A)_i$ in (10) by $\bar{\alpha}_i = (\lambda A)_i$. If a continuous variable v_j is free, we replace the constraints $c_j^+ \geq (\lambda C)_j$, $c_j^+ \geq 0$ in (10) by $0 = (\lambda C)_j$. See Section 2.5 for an explanation of why the above modifications are either necessary (in the case of free variables) or do not change the MIR closure (in the case of shifted variables).

6.2 Separation Heuristics

We next present the final cutting plane algorithm that we have implemented and describe its components.

* Strengthen bounds on variables: add MIR cuts of the form $x_i \leq \bar{\beta}$ or $x_i \geq \bar{\beta}$ for some integer $\bar{\beta}$
* Add Gomory mixed-integer cuts from the initial simplex tableau
* Repeat
  - Add MIR cuts based on formulation rows
  - Solve INT-SEP (a restriction of Appx-MIR-Sep) to find cuts based on pure integer base inequalities
  - Solve Appx-MIR-Sep with limits on the enumeration process

  Until no violated cuts are found or time is up.

6.2.1 Bound Strengthening

We take a subset S of integer variables, and for every x_i with $i \in S$, we solve LPs to maximize and minimize x_i for $x \in P^{LP}$. If $\beta_1 \leq x_i \leq \beta_2$, then $x_i \geq \lceil \beta_1 \rceil$ and $x_i \leq \lfloor \beta_2 \rfloor$ are Chvátal-Gomory cuts and therefore MIR cuts. This simple bound-strengthening procedure seems to be useful in a few MIPLIB 3.0 instances, especially **p0282**.

6.2.2 Gomory mixed-integer cuts

Gomory mixed-integer cuts for the initial LP-relaxation of the MIP are known to be MIR inequalities [24] where the multipliers used to aggregate the rows of the formulation are obtained from the inverse of the optimal basis. The base inequalities for these cuts are readily available after solving the initial relaxation and the resulting cuts are known to be effective in reducing the integrality gap significantly [5]. We use these cuts only in the first iteration of the cutting plane algorithm as the basis in the following iterations might include cuts from earlier iterations and therefore the resulting Gomory mixed-integer cuts would not necessarily be rank 1 MIR cuts, i.e., MIR cuts derived only from the constraints defining P.

As suggested by a referee, we also experiment with lift-and-project cuts, though these cuts are not generated in our default setting. In particular, we use the CglLandP cut generator [9],[2] from the COIN-OR library, which implements the Balas-Perregard [6] procedure and generates strengthened lift-and-project cuts from rows of the simplex tableau. As in the case of GMI cuts, we only invoke this cut generator in the first iteration of the cutting plane algorithm. These cuts are not used for Tables 1 and 2, but we discuss their effect separately at the end of the paper.

6.2.3 Cuts based on the rows of the formulation

Another heuristic considers rows of the formulation, one at a time, and obtains base inequalities by scaling them. Variables that have upper bounds are sometimes complemented using the bound constraints. More precisely, for a given row of the formulation and a given fractional solution, this procedure generates base inequalities by dividing the row by the coefficient of an integer variable which is currently fractional. Variables with upper bounds are complemented if their current value is closer to their upper bound than the lower bound. After writing the MIR cut, complemented variables are un-complemented to obtain a cut in the original space. This procedure was used in [20] and the authors observed that it produces effective MIR cuts.

We also note that in [24] Marchand and Wolsey describe a more sophisticated procedure that produces violated MIR inequalities by combining several rows as well as complementing variables. They observe that base inequalities obtained by combining only a few rows of the formulation can lead to effective MIR cuts. The procedure we use is motivated by their work and can be considered as a simplification of their algorithm. We noticed that even using a single row of the formulation leads to MIR inequalities that reduce the integrality gap significantly for some instances.

6.2.4 Cuts based on pure integer base inequalities

One way to generate effective MIR cuts is to concentrate on base inequalities that only contain integer variables. To obtain such base inequalities, the multiplier vector λ, used to aggregate the rows of the formulation, is required to satisfy $\lambda C \leq 0$ so that $(\lambda C)^+ = 0$. This can be achieved by fixing c^+ to zero in Appx-MIR-Sep. Note that if the original formulation has inequality constraints, the slack variables associated with these constraints are also treated as continuous variables. Therefore, multipliers associated with these rows are restricted to be non-negative for "$\geq$" constraints and non-positive for "$\leq$" constraints.

This heuristic in a way mimics the procedure to generate the so-called projected Chvátal-Gomory (pro-CG) cuts which are shown to be effective for mixed integer programs [11]. Given a multiplier vector λ such that $(\lambda C)^+ = 0$, if we denote the resulting base inequality by $\alpha x = \beta$, where $\alpha = \lambda A$ and $\beta = \lambda b$, the associated pro-CG cut is

$$\sum_{i \in I} \lceil \alpha_i \rceil x_i \;\geq\; \lceil \beta \rceil$$

and the associated MIR cut is

$$\sum_{i \in I} (\min\{\hat{\beta}, \hat{\alpha}_i\} + \hat{\beta} \lfloor \alpha_i \rfloor) x_i \;\geq\; \hat{\beta} \lceil \beta \rceil, \tag{24}$$

where $\hat{\alpha}_i$ and $\hat{\beta}$ denote the fractional part of α_i and β respectively. In other words, MIR cuts that only contain integer variables can be seen as a strengthening of pro-CG cuts.

In our implementation, we also set $\hat{\alpha}$ to zero in the separation model, and divide the objective by $\hat{\beta}$. In such a case, the objective is to maximize Δ alone. We do not then need the variables π_i or Δ_i, as we do not need to model $\hat{\beta}\Delta$. After solving this simplified model (we call this INT-SEP), we use the multipliers λ to write the cut (24). In other words, we find a violated Chvátal-Gomory (CG) cut (in the case of pure integer programs) or pro-CG cuts (in the case of mixed-integer programs), and then write the corresponding MIR cut, instead of directly finding the most violated MIR cut. The motivation for this simplification is that the resulting model was shown to be effective for pure integer programs in [23] and for mixed-integer programs in [11].

6.2.5 Cuts generated by Appx-MIR-Sep

The only parameter which must be specified for the definition and solution of Appx-MIR-Sep is the value of $\bar{k}$, i.e., the parameter responsible for the degree of approximation we use for $\hat{\beta}$. In our computational experiments, we use $\bar{k} = 5$ which is a good compromise between computational efficiency and precision. In such a way, as proved in Theorem 8, our approximate model is guaranteed to find a cut violated by at least $1/32 = .03125$.

6.3 Piloting the black-box MIP solver

A few tricks can be used to force the black-box MIP solver, in our experiments `ILOG-Cplex` 9.1, to return good heuristic solutions of both INT-SEP and Appx-MIR-Sep. Every integer feasible solution to the separation problem that has positive objective value gives a violated cut. Therefore we do not need to solve the separation problem to optimality unless we wish to claim that no violated cut exists.

To find a number of MIR cuts quickly we activate the *RINS* heuristic [19] of `ILOG-Cplex` after every 100 nodes. This approach is similar to [23] and [11]. In addition, to control the runtime in each iteration, we impose the following node limits for the enumeration tree.

- For INT-SEP, the initial node limit is set to 10,000 if no MIR cuts have been found by other heuristics, else, it is set to 1,000. After each integral solution, this limit is reset to 1,000 if the violation is less than 0.2 and 100 nodes otherwise.
- For Appx-MIR-Sep, there is no initial node limit if no MIR cuts have been found by other heuristics, else, it is set to 1,000. After each integral solution, this limit is reset to 1,000 if the violation is less than 0.1 and 100 nodes otherwise.

6.4 Computational results

In the following tables, we give our bounds for problem instances in the MIPLIB 3.0 library [8] obtained by running our algorithm with a time limit of one hour. We ignore three instances, namely **dsbmip, enigma** and **noswot** which do not have any integrality gap. In Table 1, we compare our bounds with bounds obtained for the Chvátal closure after either three or twelve hours in [23], and with the bounds obtained by Balas and Saxena, using their MIP model for split cut separation [7]. In Table 2, we compare our bounds with those obtained by 20 minutes of projected CG cuts separation in [11], and the split cut bounds from [7]. In both tables, the percentage gap closed refers to the fraction of the integrality gap closed after adding MIR cuts. Note that in [7], the bound for **arki001** is obtained by first applying the CPLEX 9.0 presolver, and then generating split cuts for the presolved problem. It is known that the CPLEX presolver can add MIR cuts or split cuts to the original model; hence the bound for **arki001** in [7] is potentially larger than the bound obtainable from its split closure.

instance	$\lvert I \rvert$	# iter	# cuts	% gap closed	time MIR	% CG gap closed	time CG	% gap split	time split
air03	10,757	1	36	100.00	1	100.0	1	100.00	3
air04	8,904	5	294	9.18	3,600	30.4	43,200	91.23	864,360
air05	7,195	12	246	12.38	3,600	35.3	43,200	61.98	24,156
cap6000	6,000	108	316	49.77	3,600	22.5	43,200	65.17	1,260
fast0507	63,009	8	318	1.66	3,600	5.3	43,200	19.08	304,331
gt2	188	96	256	98.38	2,618	91.0	10,800	98.37	599
harp2	2,993	59	523	58.48	108	49.5	43,200	46.98	7,671
l152lav	1,989	24	128	6.41	3,600	59.6	10,800	95.20	496,652
lseu	89	102	350	91.84	3,600	93.3	175	93.75	32,281
mitre	10,724	16	1126	100.00	1,396	16.2	10,800	100.00	5,330
mod008	319	54	203	98.95	201	100.0	12	99.98	85
mod010	2,655	1	39	100.00	0	100.0	1	100.00	264
nw04	87,482	91	270	93.30	3,600	100.0	509	100.00	996
p0033	33	26	110	87.42	2,552	85.3	16	87.42	429
p0201	201	254	990	74.31	3,600	60.6	10,800	74.93	31,595
p0282	282	210	1419	99.55	3,600	99.9	10,800	99.99	58,052
p0548	548	287	1317	96.11	3,600	62.4	10,800	99.42	9,968
p2756	2,756	93	671	57.57	3,600	42.6	43,200	99.90	12,673
seymour	1,372	1	559	8.35	3,600	33.0	43,200	61.52	775,116
stein27	27	123	621	0.00	3,600	0.0	521	0.00	8,163
stein45	45	539	2186	0.00	3,600	0.0	10,800	0.00	27,624

Table 1: IPs of the MIPLIB 3.0.

For a number of problems, we terminate prematurely because of numerical issues. For example, for **harp2**, after several iterations Appx-MIR-Sep returned a cut which was not violated by the point to be separated while the other separation heuristics did not return any cuts. For **p0033**, we terminate because Appx-MIR-Sep has no solution, and thus there does not exist an MIR cut which is violated by more than 1/32.

Our computed bounds are clearly sensitive to the MIP solver used and its parameter settings, e.g., if we turn off the RINS heuristic while solving Appx-MIR-Sep, for **p2756**, we get a substantially better bound, 78.19% versus only 57.57% with RINS turned on. The time saved by turning off the RINS heuristic allows our code to perform 316 iterations

instance	$\lvert I\rvert$	$\lvert J\rvert$	# iter	# cuts	% gap closed	time MIR	% CG gap closed	time CG	% gap split	time split
10teams	1,800	225	378	3334	100.00	3,600	57.14	1,200	100.00	90
arki001	538	850	18	133	33.94	3,600	28.04	1,200	83.05*	193,536
bell3a	71	62	107	404	99.60	3,600	48.10	65	65.35	102
bell5	58	46	90	629	92.95	3,600	91.73	4	91.03	2,233
blend2	264	89	510	2815	30.63	3,600	36.40	1,200	46.52	552
dano3mip	552	13,321	1	124	0.10	3,600	0.00	1,200	0.22	73,835
danoint	56	465	257	1044	1.73	3,600	0.01	1,200	8.20	147,427
dcmulti	75	473	594	3866	97.81	3,600	47.25	1,200	100.00	2,154
egout	55	86	27	264	100.00	10	81.77	7	100.00	18,179
fiber	1,254	44	105	329	94.70	3,600	4.83	1,200	99.68	163,802
fixnet6	378	500	881	4766	93.38	3,600	67.51	43	99.75	19,577
flugpl	11	7	13	28	80.23	3,600	19.19	1,200	100.00	26
gen	150	720	28	115	100.00	825	86.60	1,200	100.00	46
gesa2	408	816	494	1378	99.70	3,600	94.84	1,200	99.02	22,808
gesa2_o	720	504	448	1640	96.05	3,600	94.93	1,200	99.97	8,861
gesa3	384	768	355	892	74.83	3,600	58.96	1,200	95.81	30,591
gesa3_o	672	480	476	1382	70.82	3,600	64.53	1,200	95.20	6,530
khb05250	24	1,326	77	555	100.00	146	4.70	3	100.00	33
markshare1	50	12	5117	95369	0.00	3,600	0.00	1,200	0.00	1,330
markshare2	60	14	4580	85403	0.00	3,600	0.00	1,200	0.00	3,277
mas74	150	1	1	12	6.68	0	0.00	0	14.02	1,661
mas76	150	1	1	11	6.45	0	0.00	0	26.52	4,172
misc03	159	1	231	992	37.71	3,600	34.92	1,200	51.70	18,359
misc06	112	1,696	297	2074	99.84	792	0.00	0	100.00	229
misc07	259	1	326	1678	11.25	3,600	3.86	1,200	20.11	41,453
mod011	96	10,862	244	1673	17.41	3,600	0.00	0	72.44	86,385
modglob	98	324	1034	7060	80.04	1,677	0.00	0	92.18	1,594
mkc	5,323	2	147	4259	13.42	3,600	1.27	1,200	36.16	51,519
pk1	55	31	3988	21245	0.00	3,600	0.00	0	0.00	55
pp08a	64	176	423	1687	95.76	3,600	4.32	1,200	97.03	12,482
pp08aCUTS	64	176	611	2126	88.74	3,600	0.68	1,200	95.81	5,666
qiu	48	792	934	2142	29.19	3,600	10.71	1,200	77.51	200,354
qnet1	1,417	124	203	784	66.22	3,600	7.32	1,200	100.00	21,498
qnet1_o	1,417	124	146	587	83.78	3,600	8.61	1,200	100.00	5,312
rentacar	55	9,502	79	265	23.40	3,600	0.00	5	0.00	0
rgn	100	80	391	1142	99.60	3,600	0.00	0	100.00	222
rout	315	241	1575	9393	22.60	3,600	0.03	1,200	70.70	464,634
set1ch	240	472	179	694	76.47	3,600	51.41	34	89.74	10,768
swath	6,724	81	152	1476	33.93	3,600	7.68	1,200	28.51	2,420
vpm1	168	210	121	386	96.30	387	100.00	15	100.00	5,010
vpm2	168	210	126	427	77.71	243	62.86	1,022	81.05	6,012

Table 2: MILPs of the MIPLIB 3.0.

and generate 1550 cuts as opposed to 93 iterations with 671 cuts in Table 2. Changing the value of $\bar{k}$ also makes a difference; increasing it from 5 to 6 makes some instances of Appx-MIR-Sep harder to solve, but yielding cuts not obtainable with $\bar{k} = 5$.

Our results confirm what other authors have already noticed, i.e., that the MIR closure provides a good approximation of the optimal solution of many problems in MIPLIB 3.0. In many cases, we are able to compute bounds comparable with the ones already reported in [7, 11, 23] in a shorter computing time. In a few cases, namely **bell3a, bell5, harp2, rentacar, swath** and **gesa2**, we have been able to improve over the best bound known so far. Of course, 1 hour of CPU time to strengthen the initial formulation can be too much, but as shown in [7, 23], in a few cases such a preprocessing step allows the solution of hard unsolved problems. We believe that speeding up the MIR separation procedure would be a potentially valuable step.

To quantify the usefulness of Appx-MIR-Sep we also ran our code on the MIPLIB problems for an hour each with Appx-MIR-Sep turned off. In these runs, we used bound strengthening, GMI cuts, MIR cuts based on formulation rows, and cuts based on pure integer base inequalities returned by INT-SEP; we performed significantly more rounds of separation as solving Appx-MIR-Sep is quite time-consuming (and more than all the other methods above). In Table 3, we give the number of instances for which turning on Appx-MIR-Sep improves (worsens) the bound by more than a fixed percentage, given in the columns. For example, turning on Appx-MIR-Sep worsens the bound by more than 20% in one instance, but improves by the bound by more than 20% in 13 instances. In an extreme example, for **rgn**, we get a bound of only 6.8% if we turn off Appx-MIR-Sep, and 99.6% otherwise. On the average, 51.6% of the optimality gap was closed in these runs compared to 59.3% when Appx-MIR-Sep is activated. We also note that the average gap closed in the experiments of Balas and Saxena was 71.5% (71.3% if **arki001** is excluded), though with significantly higher computing time expended.

	percentage difference					
	1%	5%	10%	20%	50%	100%
Appx-MIR-Sep better	25	20	15	13	8	5
Appx-MIR-Sep worse	15	8	6	1	0	0

Table 3: Effect of Appx-MIR-Sep

Finally, as suggested by a referee we compare results obtained by our default setting (and given in Tables 1 and 2) with results obtained by also generating strengthened lift-and-project cuts. See Section 6.2.2. The effect of the strengthened lift-and-project cuts returned by the CglLandP cut generator is minimal, and we close only 0.4% more of the integrality gap on the average (exluding **modglob**, where we had numerical difficulties).

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