Two dimensional lattice-free cuts and asymmetric disjunctions for mixed-integer polyhedra

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Abstract

In this paper, we study the relationship between *2D lattice-free cuts*, the family of cuts obtained by taking two-row relaxations of a mixed-integer program (MIP) and applying intersection cuts based on maximal lattice-free sets in $\mathbb{R}^2$, and various types of disjunctions. Recently, Li and Richard (2007) studied disjunctive cuts obtained from t-branch split disjunctions of mixed-integer sets (these cuts generalize split cuts). Balas (2009) initiated the study of cuts for the two-row continuous group relaxation obtained from 2-branch split disjunctions. We study these cuts (and call them *cross cuts*) for the two-row continuous group relaxation, and for general MIPs. We also consider cuts obtained from asymmetric 2-branch disjunctions which we call *crooked cross* cuts. For the two-row continuous group relaxation, we show that *unimodular* cross cuts (the coefficients of the two split inequalities form a unimodular matrix) are equivalent to the cuts obtained from maximal lattice-free sets other than type 3 triangles. We also prove that all 2D lattice-free cuts and their S-free extensions are crooked cross cuts. For general mixed integer sets, we show that crooked cross cuts can be generated from a structured three-row relaxation. Finally, we show that for the corner relaxation of an MIP, every crooked cross cut is a 2D lattice-free cut.

1 Introduction

A recent topic of much interest is the generation of cutting planes for mixed-integer programs (MIPs) from canonical k-row mixed-integer sets. These canonical sets resemble the simplex tableau of a k-row MIP where all basic variables are free integer variables and all non-basic variables are nonnegative continuous variables. Clearly, these sets can be obtained simply by selecting some of the rows of the simplex tableau associated with the LP relaxation of an MIP. More generally, it is also possible to obtain a canonical k-row set as a relaxation of an MIP by first aggregating the rows of the constraint matrix of the MIP to obtain a k-row relaxation and then treating a linear combination of the original integer variables as a separate variable in each row. If the canonical set is obtained from a simplex tableau, the resulting relaxation can be viewed as a relaxation of the corner polyhedron associated with the basis defining the tableau. These relaxations are also called *k-row continuous group* relaxations. All the nontrivial valid inequalities for the canonical k-row set are intersection cuts (a concept introduced by Balas [3]) that are derived using maximal lattice-free convex sets in $\mathbb{R}^k$. We call the cutting planes for general mixed-integer sets obtained from canonical k-row sets *kD lattice-free cuts*. Gomory mixed-integer (GMI) cuts or mixed-integer rounding (MIR) cuts are 1D lattice-free cuts.

Andersen, Louveaux, Weismantel and Wolsey [1] studied two-row canonical sets in detail, and showed that the convex hull of solutions of such a set is given by split cuts and other cuts obtained from lattice-free sets in $\mathbb{R}^2$ with at most 4 sides. Subsequently, Cornuéjols and Margot [16] gave an exact characterization of the split cuts and intersection cuts based on maximal lattice-free triangles and maximal lattice-free quadrilaterals that yield facet-defining inequalities for this set. Many authors have extended these results to semi-infinite version of the canonical k-row set and to higher values of k [12] [32], to sets with more structure, such as bounds on nonbasic variables [2], the integrality of non-basic variables [20, 22], the non-negativity of basic integer variables [10, 11], [21], [25], and both the integrality of non-basic variables and non-negativity of basic integer variables [9][13]. See [18] for a recent survey on the topic.

Separately, Li and Richard [30] defined and studied cuts for mixed-integer sets obtained using t-branch split disjunctions. A t-branch split disjunction is obtained by dividing the Euclidean space into 2^t pieces based on t linearly independent splits such that any integral vector lies in exactly one of the pieces; disjunctive cuts are then obtained by taking the convex hull of the pieces intersected with the linear relaxation of the mixed-integer set. The t-branch split disjunction generalizes the standard split disjunction. Li and Richard [30] extend some results in [15] by constructing a class of mixed-integer sets with t integer variables which have facet-defining inequalities with infinite rank with respect to $(t-1)$-branch split cuts. Balas [5] proposed the study of 2-branch split cuts for the two-row continuous group relaxation.

In this paper we refer to 2-branch split disjunctions as *cross disjunctions*, and call the cuts derived from them *cross cuts*. We propose a new class of asymmetric 2-branch split disjunctions and call the resulting cuts *crooked cross cuts*. In the first half of the paper, we study the relationship between cross and crooked cross cuts and 2D lattice-free cuts, for the canonical two-row mixed-integer set. For this set, we define *unimodular cross cuts*, a subfamily of cross cuts where the coefficients of the two split inequalities form a unimodular matrix. We prove that the set of unimodular cross cuts equals the set of cuts from maximal lattice-free convex sets other than *type 3* triangles; there exist cuts from such triangles that cannot be obtained as cross cuts. Further, there exist cross cuts which cannot be obtained as unimodular cross cuts. We show that all valid inequalities (i.e., 2D lattice-free cuts) for this set are crooked cross cuts. Further, we show that some known cutting plane classes for variants of the canonical two-row set such as S-*free* cuts, or cuts which use integrality of non-basic variables (via trivial lifting or monoidal strengthening) can also be obtained as crooked cross cuts.

In the second half of the paper, we study cross cuts and crooked cross cuts for general mixed-integer sets. Nemhauser and Wolsey [29] earlier showed that split cuts, MIR cuts, and GMI cuts are equivalent (see Cornuéjols and Li [17] for a proof of the equivalence of split cuts and GMI cuts). Based on the above papers, one can easily establish the following fact: Given a split cut for a mixed-integer set obtained from a disjunction $\sum_{i=1}^n \pi_i x_i \leq \gamma \vee \sum_{i=1}^n \pi_i x_i \geq \gamma + 1$ where $\pi_i \in \mathbb{Z}$ and x_i is an integer variable for $i = 1, \ldots, n$, there is a one-row relaxation of the set, with the coefficients of x_i being π_i, such that an MIR cut derived from the relaxation is equivalent to the split cut. In other words, a split cut is a 1D lattice-free cut. Our main result is an (approximately) analogous result for cross cuts and crooked cross cuts for mixed-integer sets. We show that any (crooked) cross cut can be obtained as a (crooked) cross cut using the same disjunction from a three-row relaxation of the mixed-integer set, where the coefficients of the integer variables in two of the rows are equal to the coefficients in the inequalities defining the (crooked) cross disjunction, and zero in the third row. Further, when the coefficients of the integer variables in an MIP form a full-row rank matrix (e.g., if the set is a corner relaxation of a MIP), then (crooked) cross cuts can in fact be obtained from 2D lattice-free cuts; this generalizes Nemhauser and Wolsey's result for this class of MIPs.

The paper is organized as follows. After presenting definitions and preliminary results in Section 2, we analyze the relationship between intersection cuts using maximal lattice-free convex sets in $\mathbb{R}^2$ and cross and crooked cross cuts for the canonical two-row set in Section 3. In Section 4, we present results relating cross and crooked cross cuts with cutting planes for the canonical two-row set where more information is retained such as the integrality of non-basic variables and the non-negativity of basic integer variables. In Section 5, we present our results for general mixed integer sets. We conclude with a few open questions in Section 6.

2 Preliminaries

Consider the polyhedral mixed-integer set with m rows

$$P = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \ : \ Ax + Gy = b, \ y \geq 0\}$$

where $A \in \mathbb{Q}^{m \times n_1}$, $G \in \mathbb{Q}^{m \times n_2}$ and $b \in \mathbb{Q}^{m \times 1}$. Any mixed-integer linear program can be modeled in this way. For example, the constraint $x_i \geq 0$, where x_i is an integer variable for some i, can be replaced by the constraints $x_i - s = 0, s \geq 0$. Let P^{LP} denote the linear programming (LP) relaxation of P. We next discuss some main ingredients of disjunctive programming, introduced by Balas [4]. For convenience, vectors of coefficients in a linear inequality or equation are treated as row vectors. In particular, every π stands for a row vector.

Let $D_k \subseteq \mathbb{R}^{n_1+n_2}$ be polyhedral sets indexed by $k \in K$ with the additional property that $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \subseteq \cup_{k \in K} D_k$. We then call $D = \vee_{k \in K} D_k$ a *disjunction* and we call each D_k an *atom* of the disjunction D. A linear inequality is called a *disjunctive cut* for P obtained from the disjunction D if it is valid for $P^{LP} \cap D_k$ for all $k \in K$. All points in P satisfy any disjunctive cut for P. Note that multiple disjunctive cuts can be derived from the same disjunction. We refer to a disjunctive cut which is valid for P^{LP} as a *trivial* cut. In this paper we are interested in the following three types of disjunctions :

1. **Split disjunctions**, where $|K| = 2$ and for some $\pi \in \mathbb{Z}^{1 \times n_1}$ and $\gamma \in \mathbb{Z}^1$,
   $D_1 = \{(x, y) \in \mathbb{R}^{n_1+n_2} \ : \ \pi x \leq \gamma\}$, and,
   $D_2 = \{(x, y) \in \mathbb{R}^{n_1+n_2} \ : \ \pi x \geq \gamma + 1\}$.
2. **Cross disjunctions**, where $|K| = 4$ and for some $\pi_1, \pi_2 \in \mathbb{Z}^{1 \times n_1}$, and $\gamma_1, \gamma_2 \in \mathbb{Z}^1$,
   $D_1 = \{(x, y) \in \mathbb{R}^{n_1+n_2} \ : \ \pi_1 x \leq \gamma_1, \ \pi_2 x \leq \gamma_2\}$,
   $D_2 = \{(x, y) \in \mathbb{R}^{n_1+n_2} \ : \ \pi_1 x \leq \gamma_1, \ \pi_2 x \geq \gamma_2 + 1\}$,
   $D_3 = \{(x, y) \in \mathbb{R}^{n_1+n_2} \ : \ \pi_1 x \geq \gamma_1 + 1, \ \pi_2 x \leq \gamma_2\}$, and,
   $D_4 = \{(x, y) \in \mathbb{R}^{n_1+n_2} \ : \ \pi_1 x \geq \gamma_1 + 1, \ \pi_2 x \geq \gamma_2 + 1\}$.
3. **Crooked cross disjunctions**, where $|K| = 4$ and for some $\pi_1, \pi_2 \in \mathbb{Z}^{1 \times n_1}$, and $\gamma_1, \gamma_2 \in \mathbb{Z}^1$,
   $D_1 = \{(x, y) \in \mathbb{R}^{n_1+n_2} \ : \ \pi_1 x \leq \gamma_1, \ (\pi_2 - \pi_1)x \leq \gamma_2 - \gamma_1\}$,
   $D_2 = \{(x, y) \in \mathbb{R}^{n_1+n_2} \ : \ \pi_1 x \leq \gamma_1, \ (\pi_2 - \pi_1)x \geq \gamma_2 - \gamma_1 + 1\}$
   $D_3 = \{(x, y) \in \mathbb{R}^{n_1+n_2} \ : \ \pi_1 x \geq \gamma_1 + 1, \ \pi_2 x \leq \gamma_2\}$, and,
   $D_4 = \{(x, y) \in \mathbb{R}^{n_1+n_2} \ : \ \pi_1 x \geq \gamma_1 + 1, \ \pi_2 x \geq \gamma_2 + 1\}$.

The crooked cross disjunction is a valid disjunction since given any integer point $p \in \mathbb{Z}^{n_1}$, both $\pi_1 p$ and $\pi_2 p$ are integral. Therefore either $\pi_1 p \leq \gamma_1$ or $\pi_1 p \geq \gamma_1 + 1$. In the first case, as $(\pi_2 - \pi_1)p$ is integral, either $(\pi_2 - \pi_1)p \leq \gamma_2 - \gamma_1$ or $(\pi_2 - \pi_1)p \geq \gamma_2 - \gamma_1 + 1$. In the second case, either $\pi_2 p \leq \gamma_2$ or $\pi_2 p \geq \gamma_2 + 1$. Also note that cross and crooked cross disjunctions reduce to split disjunctions when $n_1 = 1$.

For a set $S \subseteq \mathbb{R}^n$, let $\text{int}(S)$ stand for the interior of the set. If $\vee_{k \in K} D_k$ stands for a split disjunction, then we say that the set $\mathbb{R}^{n_1+n_2} \setminus \text{int}(\cup_{k \in K} D_k)$ is a split set. We define *cross* sets and *crooked cross* sets in a similar manner. We say that a linear inequality $cx + dy \geq f$ (here $c \in \mathbb{R}^{1 \times n_1}$ and $d \in \mathbb{R}^{1 \times n_2}$) is a *split cut* for P if it is a disjunctive cut derived from a split disjunction. The *split closure* of P is the set of points in P^{LP} satisfying all split cuts for P. To obtain the closure, it suffices to consider splits where the components of π have a greatest common divisor (g.c.d.) of 1. Cook, Kannan and Schrijver [15] showed that the split closure of P is a polyhedron.

We define *cross cuts*, *crooked cross cuts*, the *cross closure*, and the *crooked cross closure* similarly. Notice that any split cut is trivially a cross cut and also a crooked cross cut. Given a split cut derived from the disjunction $D_1 \vee D_2$ as described above, a cross cut or crooked cross cut with $(\pi_1, \gamma_1) = (\pi, \gamma)$ and an arbitrary choice of (π_2, γ_2) yields the same cut because the atoms of the resulting cross or crooked cross disjunction are contained in D_1 and D_2. Consequently, the (crooked) cross closure of a set is contained in its split closure. To obtain the cross closure, it suffices to consider π_1, π_2 such that the g.c.d. of the components of each vector (denoted, e.g., by g.c.d.(π_1)) is 1. For a cross cut, if say g.c.d.$(\pi_1) \neq 1$, then the cut is dominated by another cross cut with the same π_2, and π_1 replaced by $\pi_1/$g.c.d.(π_1). We say that a cross cut or a crooked cross cut is *non-trivial* if it is not valid for the split closure of P. As discussed in Section 1, cross cuts were introduced by Li and Richard [30], who called them 2-branch split cuts.

Note that it is not immediately obvious whether the cross closure of P is contained in its crooked cross closure or vice-versa.

2.1 Cuts from 2D lattice free sets

Let $r = [r_1, r_2, ..., r_n] \in \mathbb{R}^{2\times n}$ and $f \in \mathbb{R}^2$ be such that $(f_1, f_2) \notin \mathbb{Z}^2$ and both r and f are rational. Furthermore, assume that no column of r is equal to the zero vector. In this section, we briefly review the relationship between valid inequalities for the mixed-integer set

$$W = \Big\{(z, s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ : z - rs = f\Big\} \tag{1}$$

and lattice-free convex sets in $\mathbb{R}^2$. A *lattice-free* convex set in $\mathbb{R}^2$ is one which contains no integer point in its interior. Let W^{LP} denote the continuous relaxation of W. Unless stated otherwise, by a convex set we mean a closed full-dimensional convex set. We denote the interior of a convex set B by int(B), the boundary by bnd(B), and the recession cone by rec(B).

Let B be any lattice-free convex set in $\mathbb{R}^2$ containing f in its interior. The set B can be used to generate an intersection cut [3] $\sum_{i=1}^n \alpha_i s_i \geq 1$, valid for W, where the coefficients α_i are computed as follows:

$$\alpha_i = \begin{cases} 0 & \text{if } r_i \in \text{rec}(B), \\ 1/\lambda : \lambda > 0 \text{ and } f + \lambda r_i \in \text{bnd}(B) & \text{if } r_i \notin \text{rec}(B). \end{cases} \tag{2}$$

More precisely, $\sum_{i=1}^n \alpha_i s_i \geq 1$ is a valid inequality for the set $W^{LP} \setminus (\text{int}(B) \times \mathbb{R}^n)$. (see [1].) This implies the following statement.

Remark 2.1. *Let $\vee_{k\in K} D_k$ be a disjunction in $\mathbb{Z}^2$ and assume B is contained in $\mathbb{R}^2 \setminus \text{int}(\cup_{k\in K} D_k)$, then $\sum_{i=1}^n \alpha_i s_i \geq 1$ is a disjunctive cut for W obtained from this disjunction. In particular, if B is contained in a (crooked) cross set, then $\sum_{i=1}^n \alpha_i s_i \geq 1$ is a (crooked) cross cut for W.*

Just as lattice-free sets lead to valid inequalities for W, valid inequalities for W lead to lattice-free sets. More precisely, let $\alpha s \geq 1$ be a valid inequality ($\alpha \in \mathbb{R}^{1\times n}$) for W and define $L_\alpha \subseteq \mathbb{R}^2$ as

$$L_\alpha := \{z \in \mathbb{R}^2 \ : \ \exists s \in \mathbb{R}^n_+ \text{ s.t. } \alpha s \leq 1, z = f + rs\}.$$

Andersen et. al. [1] show that L_α is a lattice-free set and $L_\alpha = conv(\{f + r_i/\alpha_i : \alpha_i > 0\}) \cup cone(\{r_i : \alpha_i = 0\})$. Furthermore, if $L_\alpha \subseteq B$ for some lattice-free convex set B, then $\alpha s \geq 1$ is implied by the intersection cut defined by B.

Definition 2.2. *A set B is called a maximal lattice-free convex set if B is lattice-free and there does not exist a convex set B' such that B' is lattice-free and $B' \supsetneq B$.*

In $\mathbb{R}^n$ any full-dimensional maximal lattice-free convex set is a polyhedral set [10, 31] with at most 2^n facets. Therefore, for $n = 2$ the maximal lattice-free convex sets are split sets, triangles, and quadrilaterals. The following more detailed classification was given by Dey and Wolsey [19]. A maximal lattice-free convex set in $\mathbb{R}^2$ is one of the following sets.

1. A split set $\{(x_1, x_2) \; : \; b \leq a_1x_1 + a_2x_2 \leq b + 1\}$ where a_1 and a_2 are coprime integers and b is an integer.

2. A triangle with a least one integral point in the relative interior of each of its sides, which in turn is either:

   (a) A type 1 triangle, i.e., a triangle with integral vertices and exactly one integral point in the relative interior of each side;

   (b) A type 2 triangle, i.e., one with at least one fractional vertex v, exactly one integral point in the relative interior of the two sides incident to v and at least two integral points on the third side;

   (c) A type 3 triangle, i.e., a triangle with exactly three integral points on the boundary, one in the relative interior of each side.

3. A quadrilateral containing exactly one integral point in the relative interior of each of its sides.

If a maximal lattice-free convex set B with f in its interior is a quadrilateral, then the cut generated using B via (2) is called a *quadrilateral cut*. Similarly, if B is a triangle of type 1, 2, or 3, the associated cut is called a *triangle cut* of type 1, 2, or 3, respectively. Andersen et. al. [1] and Cornuéjols and Margot [16] showed that the convex hull of W is given by split cuts, quadrilateral cuts, and triangle cuts.

2.2 Unimodular transformations of 2D maximal lattice-free sets

We now adapt results in [19] on lattice-free sets in "standard form". Let B be a 2D maximal lattice-free set.

If B is a quadrilateral, the four integer points on its boundary form a parallelogram of area 1. Label these points $u_1, u_2, u_3, u_4 \in \mathbb{Z}^2$ in counter-clockwise order. Then $U = (u_2 - u_1, u_4 - u_1)$ is a unimodular matrix, i.e., U has integral components and det$(U) = \pm 1$. Therefore, the mapping

$$z \to U^{-1}(z - u_1) \tag{3}$$

maps B to a maximal lattice-free set B' with $u_1 = (0, 0), u_2 = (1, 0), u_3 = (1, 1)$ and $u_4 = (0, 1)$; we say that B' is in *standard form*. See Figure 1(d). Let e_i denote the side of the quadrilateral containing u_i.

For a triangle of type 1, take any integer vertex of the triangle and the three integer points in the relative interior of the sides, and label them $u_1, \ldots, u_4$ as before, with the vertex being labeled u_1. Then U defined as before is unimodular, and the transformation in (3) maps B to a maximal lattice-free triangle of type 1 in standard form as depicted in Figure 1(a). For a triangle of type 2, any two adjacent integer points on the side containing multiple integer points along with the integer points on the other two sides form a parallelogram of area 1. Label them as before, with the adjacent points being labeled u_1 and u_4. Then the transformation in (3) maps B to a type 2 triangle in standard form, as depicted in Figure 1(b).

Finally, for a triangle of type 3, let the three integer points on the sides be (in counter-clockwise order) $u_1, u_2, u_3 \in \mathbb{Z}^2$. Define $u_4 = u_2 + u_3 - u_1$, and let $U = (u_2 - u_1, u_3 - u_1)$; U is a unimodular matrix, and the mapping in (3) maps B to a triangle of type 3, as depicted in Figure 1(c), with $u_1 = (0, 0), u_2 = (1, 0), u_3 = (0, 1)$. Denote the side that contains the integer point u_i as e_i. For triangles of type 3 in standard form, we insist that the point $(1, -1)$ lie on or below the line defining e_1; if this is not the case after applying

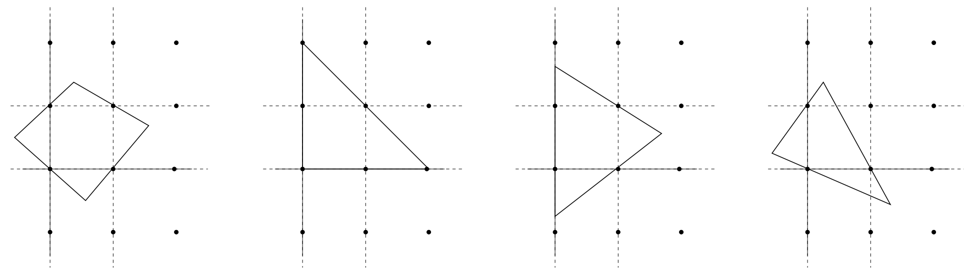


Figure 1: Lattice free sets after the transformation $U^{-1}(z - u_1)$

(3), then clearly $(-1, 1)$ lies below the line defining e_1. Then we can reflect the triangle about the line $z_1 = z_2$ (this is the same as multiplying U by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$) and bring it into standard form.

As the unimodular mapping in (3) defines a one-to-one mapping of $\mathbb{Z}^2$ to $\mathbb{Z}^2$, a half-space in $\mathbb{R}^2$ defined by $\pi x \le \gamma$ with π, γ integral, is mapped to another half-space $\pi' x \le \gamma'$ with π', γ' integral. Therefore a cross set is mapped to a cross set with this mapping.

Remark 2.3. *If a maximal lattice-free set B is mapped to B' with the mapping in (3), and B' is contained in a cross set, then applying the inverse transformation $z \to Uz + u_1$ we obtain the fact that B is contained in a cross set. Analogous statements can be made for split sets and crooked cross sets.*

2.3 Cuts for general mixed-integer sets from 2D lattice-free cuts

One can use 2D lattice-free convex sets to obtain cuts for general MIPs in the following manner. A *general 2D lattice-free cut* for the mixed-integer set P is an inequality $\alpha y \ge 1$ which can be obtained as a quadrilateral cut or triangle cut for the following two-row relaxation of P:

$$P_2(\lambda_1, \lambda_2) = \{(z, y) \in \mathbb{Z}^2 \times \mathbb{R}^{n_1} \ : \ z_1 + g_1 y = b_1, \ z_2 + g_2 y = b_2, \ y \ge 0\},$$

where $g_i = \lambda^i G$, $b_i = \lambda^i b$ for some $\lambda^i \in \mathbb{R}^{1 \times m}$ that satisfies $\lambda^i A \in \mathbb{Z}^{n_1}$, for $i = 1, 2$. Notice that we can take the canonical 2-row set W and using integer row vectors $\lambda_1, \lambda_2 \in \mathbb{Z}^2$ obtain the set $W_2(\lambda_1, \lambda_2)$. In this case, the quadrilateral cuts for $W_2(\lambda_1, \lambda_2)$ give valid inequalities for W but these cuts are not necessarily quadrilateral cuts for W. We discuss this further in Section 3.3.

3 Lattice-free cuts as disjunctive cuts for canonical two row sets

In this section, we study the relationship between lattice-free cuts and (crooked) cross cuts for the two-row canonical set W. Consider a cross disjunction defined by $\pi_1, \pi_2 \in \mathbb{Z}^2$ and $\gamma_1, \gamma_2 \in \mathbb{Z}$. We say that a cross cut is a *unimodular cross cut* if the matrix $[\pi_1^T \ \pi_2^T]$ is unimodular and a *non-unimodular cross cut* otherwise. We distinguish between unimodular and non-unimodular cross disjunctions and sets similarly. Note that the set of unimodular cross sets is invariant under unimodular transformations of the type (3). We call the set of points in W^{LP} which satisfy all unimodular cross cuts for W its *unimodular cross closure*. We denote the unimodular cross closure, the cross closure, and the crooked cross closure by $\mathcal{UC}, \mathcal{C}, \mathcal{CC}$ respectively. Recall from Section 2 that the split closure of W contains both $\mathcal{C}$ and $\mathcal{CC}$. A similar result holds for $\mathcal{UC}$. To see this, consider a split cut defined by $\pi, \gamma \in \mathbb{Z}^2 \times \mathbb{Z}$ as in Section 2 with atoms D_1, D_2. Recall that the components of π can be assumed to have a g.c.d. of one. Let $(\pi_1, \gamma_1) = (\pi, \gamma)$. Choose a vector π_2 in $\mathbb{Z}^2$ such that the matrix $[\pi_1^T \ \pi_2^T]$ is unimodular; this exists because g.c.d.(π_1) is one. Finally, let γ_2 be an arbitrary integer. Then the atoms of the resulting unimodular cross disjunction are contained in D_1 and D_2, and the split cut is a unimodular cross cut.

We call the set of points in W^{LP} which satisfy all quadrilateral cuts for W its *quadrilateral closure*, and denote it by $\mathcal{Q}$. We define $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ to denote points in W^{LP} that satisfy all triangle cuts of types 1,2, or 3, respectively. Basu et. al [7, Theorem 1.4] showed that $\mathcal{Q}$ is contained in the split closure of W, and also (see Figure 6 in their paper) that $\mathcal{Q} \subseteq \mathcal{T}_1, \mathcal{T}_2$. Therefore the intersection $\mathcal{Q} \cap \mathcal{T}_3$ equals the convex hull of W.

In this section we establish the following relationships between these closures.

Theorem 3.1. *For the 2 row canonical set W the following holds: (i) $\mathcal{Q} = \mathcal{UC}$ and (ii) $\mathcal{CC} = \mathcal{Q} \cap \mathcal{T}_3$. Therefore $\mathcal{Q} = \mathcal{UC} \supseteq \mathcal{C} \supseteq \mathcal{CC} = \mathcal{Q} \cap \mathcal{T}_3$.*

Note that the inclusion $\mathcal{UC} \supseteq \mathcal{C}$ trivially follows from the definition of the corresponding sets and the inclusion $\mathcal{C} \supseteq \mathcal{CC}$ follows from the fact that $\mathcal{C} \supseteq \mathcal{Q} \cap \mathcal{T}_3$ and $\mathcal{CC} = \mathcal{Q} \cap \mathcal{T}_3$.

3.1 Obtaining 2D lattice-free cuts as disjunctive cuts

We begin by showing that $\mathcal{Q} \supseteq \mathcal{UC}$.

Lemma 3.2. *A quadrilateral cut for W or a triangle cut of type 1 or 2 is a unimodular cross cut for W. Consequently, $\mathcal{Q} \supseteq \mathcal{UC}$.*

Proof. Let B be an instance of one of the classes of maximal lattice-free convex sets mentioned in the Lemma, and assume it is in standard form, as depicted in Figure 1. If B is a type 1 triangle, then B is the triangle with vertices $(0, 0)$, $(2, 0)$, $(0, 2)$ and is contained in the unimodular cross set

$$\{z \in \mathbb{R}^2 \ : \ 0 \le z_1 \le 1\} \cup \{z \in \mathbb{R}^2 \ : \ 0 \le z_2 \le 1\}. \tag{4}$$

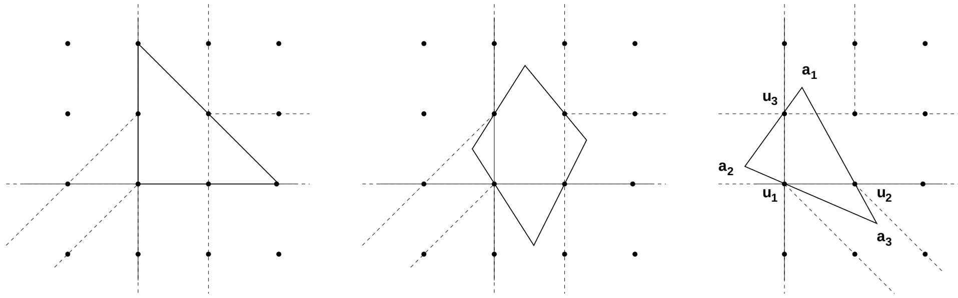
In a similar manner, if B is a type 2 triangle or a maximal quadrilateral in standard form, it is contained in the cross set in (4). The result follows from Remarks 2.3 and 2.1, and from the fact that unimodular cross sets are transformed to unimodular cross sets by unimodular transformations. ■

The next result and its corollary show that there exists triangle cuts of type 3 which are not unimodular cross cuts, nor even cross cuts.

Lemma 3.3. *No maximal lattice-free triangle of type 3 is contained in a cross set.*

Proof. Let B be a type 3 triangle in standard form. Let the vertices of B be a_1, a_2, a_3, where a_i is the vertex opposite the side e_i, for $i = 1, 2, 3$. We will show that there do not exist $\pi_1, \pi_2 \in \mathbb{Z}^2$ and $\gamma_1, \gamma_2 \in \mathbb{Z}$ such that B is contained in the cross set $\{z \in \mathbb{R}^2 \ : \ \gamma_1 \le \pi_1 z \le \gamma_1 + 1\} \cup \{z \in \mathbb{R}^2 \ : \ \gamma_2 \le \pi_2 z \le \gamma_2 + 1\}$.

Assume by contradiction that there are two split sets whose union contains B. Then these split sets contain all the three vertices. Therefore, at least two of them must belong to one of the split sets. Consider first the case where a_2 and a_3 belong to the same split set. Since there is an integer point in the relative interior of the side e_1, the side joining a_2 to a_3, one facet of the split set coincides with e_1. Let this facet be defined by the line $\{z \in \mathbb{R}^2 : \alpha_1 z_1 + \alpha_2 z_2 = 0\}$ where $\alpha_1, \alpha_2 \in \mathbb{Z}_+$. Note that $\alpha_1 < \alpha_2$ as the point $(1, -1)$ lies below this line. Therefore the other facet of the split set is of the form $\{z \in \mathbb{R}^2 \ : \ \alpha_1 x_1 + \alpha_2 x_2 = \alpha'\}$ where $0 < \alpha' \le \alpha_1$; if $\alpha' > \alpha_1$, then the point $(1, 0)$ lies in the interior of the split set. Consider the largest



(a) (b) (c)

Figure 2: Crooked cross sets containing maximal quadrilaterals and triangles

possible split set $\{z \in \mathbb{R}^2 : 0 \leq \alpha_1 z_1 + \alpha_2 z_2 \leq \alpha_1\}$. Since $\alpha_1 < \alpha_2$, the line $\{z \in \mathbb{R}^2 : \alpha_1 z_1 + \alpha_2 z_2 = \alpha_1\}$ lies below the point $(0, 1)$ and thus intersects the side e_3 at a point p whose z_2 coordinate is strictly less than 1. Therefore, the point $(0, 1)$ lies in the relative interior of the line segment pa_1.

For the cross set to contain B, the second split set must contain the triangle with vertices $p, a_1, (1, 0)$. Therefore, one of the facets of the second split set must be along the line segment pa_1. Let this facet be $\{z \in \mathbb{R}^2 \ : \ \beta_1 z_1 + \beta_2 z_2 = \beta_2\}$ where $\beta_1 < 0$, $\beta_2 > 0$ and $\beta_1, \beta_2 \in \mathbb{Z}$. The other facet of this split set is defined by a line $\{z \in \mathbb{R}^2 \ : \ \beta_1 z_1 + \beta_2 z_2 = \beta'\}$ where $\beta' \geq 0$. The largest possible split set would be when the facet has the form $\{z \in \mathbb{R}^2 \ : \ \beta_1 z_1 + \beta_2 z_2 = 0\}$. As $(1, 0)$ lies below this line, it intersects e_2 at a point q whose z_2 coordinate is strictly positive. Thus, the boundary of B between q and $(1, 0)$ is not contained in any of the splits, a contradiction.

The case where a_1 and a_3 belong to the same split set or a_1 and a_3 belong to the same split set can be similarly analyzed. ∎

Corollary 3.4. *There exists a type 3 triangle cut for W that is not a cross cut.*

Proof. Consider a set W with three continuous variables, and a maximal lattice-free triangle B of type 3 such that $f + r_i$ ($i \in \{1, 2, 3\}$) are the three vertices of B. Then the inequality $\alpha s \geq 1$, where $\alpha = (1, 1, 1)$, is generated using B via (2). We will show that this inequality is not a cross cut. It is clear that for any point z^* in int(B) there exists an $s^* \geq 0$ such that $z^* = f + rs^*$ and $\alpha s^* < 1$. Now if $\alpha s \geq 1$ is a cross cut, then there exists a cross disjunction $C \times \mathbb{R}^n$ such that C is a cross disjunction in $\mathbb{R}^2$, and $\alpha s \geq 1$ is valid for the intersection of W^{LP} with each atom of the disjunction. By Lemma 3.3, one of the atoms of C (say $\mathcal{A}$) contains a point z^* in int(B). Then, there exists a point (z^*, s^*) in $W^{LP} \cap (\mathcal{A} \times \mathbb{R}^n)$ with $\alpha s^* < 1$, a contradiction. ∎

While Corollary 3.4 shows that some type 3 triangle cuts are not cross cuts, this does not imply that $\mathcal{C} \not\subseteq \mathcal{T}_3$. To illustrate this point, consider the case of a triangle cut of type 1 that is not dominated by a single quadrilateral cut. However, the triangle cut is valid for $\mathcal{Q}$. This follows from the fact that for any triangle cut of type 1, one can construct (see Basu et. al [7]) an infinite sequence of quadrilateral cuts which in a formal sense converge to the triangle cut.

Next we show that unlike cross cuts, all the triangle and quadrilateral inequalities can be obtained in a very simple way by using crooked cross cuts. For type 3 triangles, some elements of the proof can be found in [19].

Lemma 3.5. *Any quadrilateral cut or triangle cut for W is a crooked cross cut for W. Consequently, $\mathcal{CC} \subseteq \mathcal{Q} \cap \mathcal{T}_3$.*

Proof. We will show that every maximal lattice-free set B in $\mathbb{R}^2$ in standard form is contained in a crooked cross set. This is obvious for triangles of type 1 or 2 in Figure 1; they are both contained in the set

$$\mathcal{C}_1 = \{z : 0 \le z_1 \le 1\} \cup \{z : z_1 \ge 1 \text{ and } 0 \le z_2 \le 1\} \cup \{z : z_1 \le 0 \text{ and } 0 \le z_2 - z_1 \le 1\}.$$

See Figure 2(a) for a depiction of $\mathcal{C}_1$ (by the dashed lines).

Now consider a maximal lattice-free quadrilateral Q in standard form. In Section 2.2 we denoted the integral points on the boundary of Q to be $u_1, \dots, u_4$, which are $(0,0), (1,0), (0,1), (1,1)$, respectively, and also defined the edge containing u_i to be e_i, for $i = 1, \dots, 4$. It is well-known that Q is contained in the cross set $\mathcal{C}_2 = \{z : 0 \le z_1 \le 1\} \cup \{z : 0 \le z_2 \le 1\}$. To see this, note that for any point v in $\mathbb{R}^2 \setminus \mathcal{C}_2$, the convex hull of v and $u_1, \dots, u_4$ is a convex body containing one of $u_1, \dots, u_4$ in its interior; for example, if $v \in \{z : z_1 < 0, z_2 < 0\}$, then u_1 lies in the interior of the convex hull of v and $u_1, \dots, u_4$.

As Q is a quadrilateral, the angle between the sides incident with some vertex is 90 degrees or more. Without loss of generality we can assume this vertex is the one incident with the edges e_1 and e_4 (if this condition does not hold, we can rotate the body via a unimodular transformation to attain it). The side incident with this vertex having positive slope is e_4, and the other side is e_1. One of these two sides forms an angle of at most 45 degrees with the line segment joining (0,0) and (0,1); assume it is e_4. Therefore $z_2 - z_1 \le 1$ is a valid inequality for $Q \cap \{z : z_1 \le 0\}$. As $Q \subseteq \mathcal{C}_2$, $z_2 \ge 0$ is a valid inequality for $Q \cap \{z : z_1 \le 0\}$, and so is $z_2 - z_1 \ge 0$. Therefore $Q \cap \{z : z_1 \le 0\} \subseteq \mathcal{C}_1$. Clearly $\mathcal{C}_1 \cap \{z : z_1 \ge 0\} = \mathcal{C}_2 \cap \{z : z_1 \ge 0\}$. Therefore Q is contained in the crooked cross set $\mathcal{C}_1$.

Let B be a maximal lattice-free triangle of type 3 in standard form that contains the points $u_1 = (0,0)$, $u_2 = (1,0)$ and $u_3 = (0,1)$ in the relative interior of its sides. Denote the vertex that is opposite to the edge that contains $u_1 = (0,0)$ by a_1, the vertex that is opposite to the edge that contains $u_2 = (1,0)$ by a_2 and the vertex that is opposite to the edge that contains $u_3 = (0,1)$ by a_3. Let $T = \mathbb{R}^2 \setminus (S_1 \cup S_2 \cup S_3)$ where

$$S_1 = \{z : 0 \le z_1 \le 1\}, \quad S_2 = \{z : 0 \le z_2 \le 1\}, \quad S_3 = \{z : 0 \le z_1 + z_2 \le 1\}.$$

Furthermore, let these sets be partitioned into smaller closed components as shown in Figure 3. (For example $S_1 = S_1^+ \cup V_1 \cup V_0 \cup V_3 \cup S_1^-$.) By elementary geometry, note that the point a_1, which is the vertex opposite to the edge containing point $u_1 = (0,0)$ must lie in the cone generated by rays $\overrightarrow{u_1u_2}$ and $\overrightarrow{u_1u_3}$ and furthermore, it can not lie inside the triangle obtained by taking the convex hull of the points u_1, u_2 and u_3. Consequently, $a_1 \in S_1^+ \cup V_1 \cup S_2^+ \cup T_1$. Similarly, $a_2 \in S_2^- \cup V_2 \cup S_3^- \cup T_3$ and $a_3 \in S_1^- \cup V_3 \cup S_3^+ \cup T_5$.

Notice that B can not intersect with the interior of T_1 as otherwise $(1,1)$ would belong to the interior of B. Similarly, the intersection of B with the interior of T_2, T_3, T_4, T_5 and T_6 is empty. Therefore, B and all three vertices of B lie in $S_1 \cup S_2 \cup S_3$. We will next show that B does not intersect with the interiors of the sets S_1^-, S_2^+ and S_3^- and conclude that B in fact looks like the triangle shown in Figure 2(c).

As B is in standard form, we know that a_3 satisfies the inequality $z_1 + z_2 \ge 0$ (see Section 2.2) and therefore $a_3 \notin \text{int}(S_1^-)$ and therefore $a_3 \in V_3 \cup S_3^+$. In addition, as $u_1 = (0,0)$ is a convex combination of points a_2 and a_3, the point a_2 must satisfy $z_1 + z_2 \le 0$. Therefore, we conclude that $a_2 \in S_2^-$.

The point $u_3 = (0,1)$ is a convex combination of a_1 and a_2 and as a_2 satisfies $z_2 \le 1$, the point a_1 must satisfy $z_2 \ge 1$. Furthermore, as $a_1 \in S_1^+ \cup U_1 \cup S_2^+ \cup T_1$ and $B \cap \text{int}(T_1) = \emptyset$, we conclude that $a_1 \in S_1^+$.

Clearly, the edge connecting $a_1 \in S_1^+$ and $a_2 \in S_2^-$ and going through $u_3 = (0,1)$ does not intersect the interior of S_3^-. Similarly, the edge connecting $a_1 \in S_1^+$ and $a_3 \in V_3 \cup S_3^+$ and going through $u_2 = (1,0)$ does not intersect the interior of S_2^+, and, the edge connecting $a_2 \in S_2^-$ and $a_3 \in V_3 \cup S_3^+$ and going through $u_1 = (0,0)$ does not intersect the interior of S_1^-.

Therefore, we conclude that $B \cap S_2^+ = \{u_2\}$, $B \cap S_1^- = \{u_1\}$ and $B \cap S_3^- = \{u_3\}$ implying that

$$\mathcal{C}_3 = \{z : z_1 \leq 0 \text{ and } 0 \leq z_2 \leq 1\} \cup \{z : z_2 \geq 0 \text{ and } 0 \leq z_1 \leq 1\} \cup \{z : z_2 \leq 0 \text{ and } 0 \leq z_2 + z_1 \leq 1\}$$

contains B. However, $\mathcal{C}_3$ is a subset of the crooked cross set contained within the dashed lines in Figure 2(c) (defined by removing $z_1 \leq 0$ from the first set in $\mathcal{C}_3$, and changing $z_2 \geq 0$ to $z_2 \geq 1$ in the second set). ■

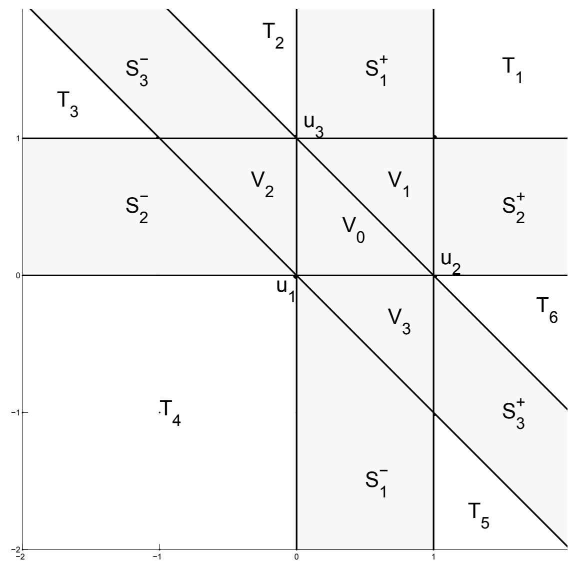


Figure 3: Partitioning of $\mathbb{R}^2$ by the split sets S_1, S_2 and S_3

Recall that Andersen et. al. [1] and Cornuéjols and Margot [16] proved that $conv(W)$ is given by all quadrilateral cuts, triangle cuts and split cuts. Therefore Lemma 3.5 implies that $\mathcal{CC} = \mathcal{Q} \cap \mathcal{T}_3$.

To complete the proof of Theorem 3.1, we will show in Section 3.2 that $\mathcal{Q} \subseteq \mathcal{UC}$.

3.2 Obtaining disjunctive cuts as 2D lattice-free cuts

Andersen et. al.[1, Corollary 1] show that if W is non-empty and $\alpha s \geq \alpha_0$ is a facet-defining inequality for W, then $\alpha \geq 0$. It is easy to see that this property holds for any valid inequality, not just facet-defining ones. We use this fact in proving the following observation.

Lemma 3.6. *If W is the empty set, i.e., W has no integral solutions, then the split closure of W equals $\emptyset$.*

Proof. Let W be the empty set. If r has two linearly independent columns, say r_1 and r_2, then the set $\{z \in \mathbb{R}^2 : z = f + r_1 s_1 + r_2 s_2,\ s_1, s_2 \geq 0\}$ contains integer points, and W is non-empty, a contradiction. Therefore, r must have rank 1. Let $W_z := \{z \in \mathbb{R}^2 : z = f + rs \text{ for some } s \in \mathbb{R}^n_+\}$. Then W_z is either a half-line or a line in $\mathbb{R}^2$, and points in W_z satisfy $a_1 z_1 + a_2 z_2 = b$, for some co-prime integers a_1 and a_2. Assume b is integral. Then there is an integer point $z^* \in \mathbb{Z}^2$ satisfying $a_1 z_1^* + a_2 z_2^* = b$. This means that if W_z is a line, it is non-empty, a contradiction. If W_z is a half-line, then there is a large enough integer $t > 0$ such that tr_1 is integral and $z^* + tr_1$ lies in W_z. Therefore b must be non-integral. Then W has empty intersection with each side of the disjunction

$$(a_1 z_1 + a_2 z_2 \leq \lfloor b \rfloor) \vee (a_1 z_1 + a_2 z_2 \geq \lceil b \rceil),$$

and the split closure of W is the empty set. ■

Some elements of the proof of the next result were observed by Balas [5]. In particular, he observed that given a derivation of a cut using a cross disjunction, one can use the multipliers for the inequalities defining the cross set to obtain a quadrilateral which may be a maximal lattice-free quadrilateral or (in degenerate cases) a maximal lattice free triangle of type 1 or 2.

For the purpose of the next Proposition, we define the vertex of an atom as the point where the defining inequalities of the atom intersect, i.e., the vertex of the atom $\pi_1 z \le \gamma_1, \pi_2 z \le \gamma_2$ is the point z^* satisfying $\pi_1 z^* = \gamma_1, \pi_2 z^* = \gamma_2$.

Proposition 3.7. *Any non-trivial unimodular cross cut for W is implied by either a quadrilateral cut or a triangle cut of type 1 or 2.*

Proof. Consider a non-trivial unimodular cross cut for W. The existence of such a cut implies that W is non-empty,

otherwise by Lemma 3.6 the split closure of W is the empty set, and therefore so is the cross closure of W, a contradiction. We will next derive four inequalities defining a lattice-free set from the multipliers used in obtaining the cross cut as a valid inequality in each atom of the disjunction. This set will be contained in a maximal lattice-free quadrilateral or triangle of type 1 or 2.

Let the cross cut be $cz + gs \ge d$, where $(c, g) \in \mathbb{R}^{1\times(2+n)}$, and the cut is derived from the disjunction $\vee_{i=1}^4 D_i$ where the sets D_i are defined as in Section 2 via the inequalities

$$\pi_1 z \le \gamma_1,\ \pi_1 z \ge \gamma_1 + 1,\ \pi_2 z \le \gamma_2,\ \pi_2 z \ge \gamma_2 + 1.$$

By subtracting appropriate multiples of the constraints $z - rs = f$ from the cut $cz + gs \ge d$, we can assume that it has the form $\alpha s \ge \alpha_0$. As W is non-empty, we can assume that $\alpha \ge 0$.

As $\alpha s \ge 0$ is valid for W^{LP}, but the cross cut is not, we can assume $\alpha_0 > 0$. Therefore, we can assume, without loss of generality, that the cross cut has the form $\alpha s \ge 1$. We next define a lattice-free convex set Q_α such that the cut defined by (2) implies $\alpha s \ge 1$. More precisely, we will show that $L_\alpha \subseteq Q_\alpha$.

Consider the atom D_1 of the disjunction given by $\pi_1 z \le \gamma_1, \pi_2 z \le \gamma_2$. Consider the case when the intersection of this atom with W^{LP} is non-empty. By definition, $\alpha s \ge 1$ is valid for

$$\begin{aligned} & z - rs = f, s \ge 0, && (5)\\ & -\pi_1 z \ge -\gamma_1, \\ & -\pi_2 z \ge -\gamma_2. \end{aligned}$$

Using LP duality, there exists multipliers $\mu \in \mathbb{R}^{1\times 2}$ (for $z - rs = f$), and $v, w \ge 0$, such that

$$\mu - v\pi_1 - w\pi_2 = 0, \tag{6}$$

$$-\mu r \le \alpha, \tag{7}$$

$$\mu f - v\gamma_1 - w\gamma_2 \ge 1. \tag{8}$$

Equation (6) follows from the fact that the z variables are free, and (7) from the fact that $s \ge 0$. Therefore $\mu = v\pi_1 + w\pi_2$. Note that as $\alpha s \ge 1$ is not valid for W^{LP}, one of v, w must be positive.

We define one side of Q_α corresponding to D_1 to be $\mu z \ge v\gamma_1 + w\gamma_2$. We next show that L_α satisfies the above inequality, but no integer point in D_1 does. The inequality (8) implies that f strictly satisfies the inequality $\mu z \ge v\gamma_1 + w\gamma_2$. Further, for each $\alpha_i > 0$, this inequality is satisfied by $(f + r_i/\alpha_i)$ as

$$\begin{aligned} \alpha_i \ge -\mu r_i \quad &\Rightarrow \quad \alpha_i \ge -\mu r_i/(\mu f - v\gamma_1 - w\gamma_2) \ \Rightarrow \\ \mu f - v\gamma_1 - w\gamma_2 \ge -\mu r_i/\alpha_i \quad &\Rightarrow \quad \mu(f + r_i/\alpha_i) \ge v\gamma_1 + w\gamma_2. \end{aligned}$$

If $\alpha_i = 0$, then this inequality is satisfied by $f + \lambda r_i$ for all $\lambda \geq 0$: as $-\mu r_i \leq 0$ or $\mu r_i \geq 0$ by (7), it follows that

$$\mu(f + \lambda r_i) - v\gamma_1 - w\gamma_2 \quad \geq \quad \mu f - v\gamma_1 - w\gamma_2 \; \geq \; 1.$$

Therefore r_i is a direction in the recession cone of the set $\{z \in \mathbb{R}^2 \; : \; \mu z \geq v\gamma_1 + w\gamma_2\}$ (since $\{z \in \mathbb{R}^2 \; : \; \mu z \geq v\gamma_1 + w\gamma_2\}$ is a closed set). Hence L_α is contained in the half-space defined by

$$(v\pi_1 + w\pi_2)z \geq v\gamma_1 + w\gamma_2. \tag{9}$$

Further, any integer vector $\bar{z}$ contained in the atom D_1 satisfies $(v\pi_1 + w\pi_2)\bar{z} \leq v\gamma_1 + w\gamma_2$ as $v, w \geq 0$ and one of v, w is positive. Therefore, $\bar{z}$ cannot lie in the interior of the half-space (9). Finally, if z^* is a unique vertex of one of the other atoms, then $\pi_1 z^* \geq \gamma_1$ and $\pi_2 z^* \geq \gamma_2$ with one of these two inequalities holding as a strict inequality. Therefore z^* also satisfies (9).

If the intersection of D_1 with W^{LP} is empty, then $0s \geq 1$ is implied by the constraints in (5). Therefore, there exist multipliers $\mu \in \mathbb{R}^2$ and $v, w \geq 0$ satisfying (6), (8) and $-\mu r \leq 0$ instead of $-\mu r \leq \alpha$. Then $\mu r \geq 0 \Rightarrow \mu r_i/\alpha_i \geq 0$ for every positive α_i. Therefore $\mu f > v\gamma_1 + w\gamma_2$ implies that $\mu(f + r_i/\alpha_i) > v\gamma_1 + w\gamma_2$. Similar to the previous case, if $\alpha_i = 0$, then we can verify that r_i is a recession direction for (9). Therefore L_α satisfies the inequality (9), and f lies in the interior of the corresponding half-space. Moreover, it can again be verified that all the integer points $\bar{z}$ belonging to D_1 satisfy $(v\pi_1 + w\pi_2)\bar{z} \leq v\gamma_1 + w\gamma_2$ and if z^* is a unique vertex of one of the other atoms, then $(v\pi_1 + w\pi_2)z^* \geq v\gamma_1 + w\gamma_2$.

As the choice of the atom is arbitrary, we can similarly derive inequalities (9) for each atom, and assert that f is contained in the interior of the set Q_α defined by these inequalities, and that L_α is contained in Q_α. Further, any integer point in $\mathbb{R}^2$ is contained in one of the atoms of the disjunction, and cannot therefore be contained in the interior of Q_α. Therefore Q_α is lattice-free, and the intersection cut defined by it implies the cross cut $\alpha s \geq 1$.

We will now show that Q_α is contained in a maximal lattice-free quadrilateral or triangle of type 1 or 2. By definition, Q_α has at most 4 sides. Further, the unique vertex of each atom of the cross disjunction is integral. For example, consider the atom $\{z \in \mathbb{R}^2 : \pi_1 z \leq \gamma_1, \pi_2 z \leq \gamma_2\}$; its unique vertex is the point z^* satisfying $\pi_1 z^* = \gamma_1$ and $\pi_2 z^* = \gamma_2$. As the matrix $\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}$ is unimodular, z^* is integral. These integral vertices of the atoms all lie in Q_α and therefore must lie on its boundary. Now Q_α is full-dimensional as it contains f in its interior.

If it is not maximal, then consider a maximal lattice-free convex set B containing it; B must also contain the integer vertices of the atoms on its boundary. As there are 4 such points, B cannot be a triangle of type 3. Further it cannot be a split set, as that would contradict the non-triviality of $cz + gs \geq d$. Therefore, B must be a quadrilateral, or a triangle of type 1 or 2. ∎

In Figure 4, we depict two linearly independent splits defining a disjunction, the atom D_1 and the inequality $(v\pi_1 + w\pi_2)z \geq v\gamma_1 + w\gamma_2$ by the dashed line.

Corollary 3.8. $\mathcal{Q} \subseteq \mathcal{UC}$.

Proof. If there does not exist a non-trivial unimodular cross cut, then $\mathcal{UC}$ equals the split closure of W, but $\mathcal{Q}$ is contained in the split closure of W as shown by Basu et. al. [7], and the result follows. Assume there exists a non-trivial unimodular cross cut. The previous result implies that the unimodular cross cut is implied by a quadrilateral cut or a triangle cut of type 1 or 2. Therefore $\mathcal{Q} = \mathcal{Q} \cap \mathcal{T}_1 \cap \mathcal{T}_2 \subseteq \mathcal{UC}$. ∎

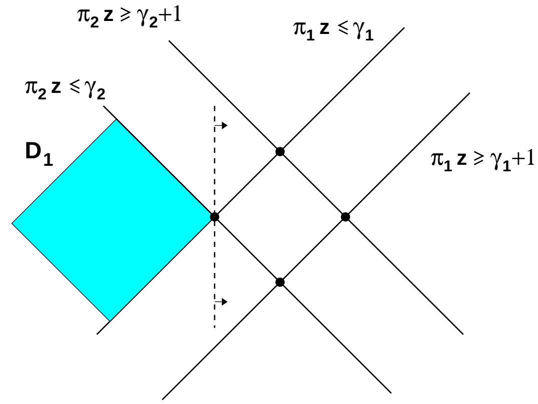


Figure 4: The disjunction defining a cross cut

3.3 Relative strength of unimodular and non-unimodular cross cuts

It is not obvious if it is possible to obtain non-unimodular cross cuts that are not implied by any (or a combination of) unimodular cross cuts. We next present a non-unimodular cross cut which is not dominated by any single unimodular cross cut.

Example 3.9. *Let W^* be the set of points $(z, s) \in \mathbb{Z}^2 \times \mathbb{R}^4_+$ that satisfy*

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} s_1 + \begin{pmatrix} -1 \\ 1 \end{pmatrix} s_2 + \begin{pmatrix} 11 \\ -6 \end{pmatrix} s_3 + \begin{pmatrix} 1 \\ -2 \end{pmatrix} s_4. \tag{10}$$

Now consider the non-unimodular cross set

$$\{(z_1, z_2) \in \mathbb{R}^2 \ : \ 0 \leq z_1 + z_2 \leq 1\} \cup \{(z_1, z_2) \in \mathbb{R}^2 \ : \ 0 \leq z_1 - z_2 \leq 1\}. \tag{11}$$

One can show that the inequality $s_1 + s_2 + 14s_3 + 2s_4 \geq 1$ is implied by the cross disjunction associated with the above cross set. To this end, let $Q := \text{conv}\{f + \frac{r_i}{\alpha_i}\}$, where $\alpha = (1, 1, 14, 2)$, f is the constant vector in (10), and $r_1, \ldots, r_4$ are the columns associated with the variables $s_1, \ldots, s_4$ in (10). Thus Q is the quadrilateral with vertices $(\frac{1}{4}, -\frac{1}{2})$, $(-\frac{3}{4}, \frac{3}{2})$, $(\frac{29}{28}, \frac{1}{14})$, and $(\frac{3}{4}, -\frac{1}{2})$, and is depicted by the shaded object in Figure 5. One can verify that Q is contained in the cross set (11); in particular, the sides of Q pass through the vertices of the atoms of the cross disjunction.

Proposition 3.10. *The inequality $s_1 + s_2 + 14s_3 + 2s_4 \geq 1$ is not a unimodular cross cut for W^*.*

Proof. If the inequality $s_1 + s_2 + 14s_3 + 2s_4 \geq 1$ is a unimodular cross cut for W^*, then by Proposition 3.7, there exists a maximal lattice-free quadrilateral or a maximal lattice-free triangle of type 1 or type 2 that contains Q. We will however show that any maximal lattice-free convex set containing the set Q is a maximal lattice-free triangle of type 3. This would therefore imply that the inequality $s_1 + s_2 + 14s_3 + 2s_4 \geq 1$ is not a unimodular cross cut for W^*.

Let M be the maximal lattice-free convex set containing Q. Two facet-defining inequalities of M are $2z_1 + z_2 \geq 0$ and $-2z_1 + z_2 \geq -2$: both the lines $\{(z_1, z_2) \,|\, 2z_1 + z_2 = 0\}$ and $\{(z_1, z_2) \,|\, -2z_1 + z_2 = -2\}$ define facets of Q and contain integer points in their relative interior. Therefore, these must be facets of M.

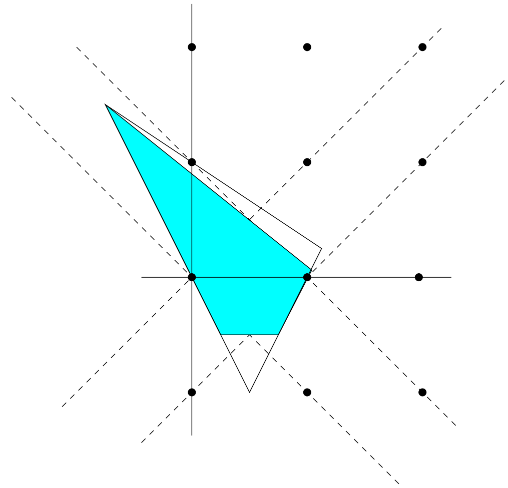


Figure 5: Example of lattice-free convex set that is contained in a non-unimodular cross set and not in a unimodular cross set.

The intersection of the lines $2z_1 + z_2 = 0$ and $-2z_1 + z_2 = -2$ (i.e., $(\frac{1}{2}, -1)$) is a vertex of M. This is because M must contain an integer point in the relative interior of each of its facets and while the facets of M defined by the lines $2z_1 + z_2 = 0$ and $-2z_1 + z_2 = -2$ contain the integer points $(0, 0)$ and $(1, 0)$, respectively, the convex hull of $(0, 0)$, $(1, 0)$, $(\frac{1}{2}, -1)$ contains no integer points in its interior.

Since $2z_1 + z_2 \geq 0$ and $-2z_1 + z_2 \geq -2$ define facets of M, all potential candidates for integer points on the boundary of M are integer points satisfying these inequalities. Note that $(-\frac{3}{4}, \frac{3}{2})$ and $(\frac{29}{28}, \frac{1}{14})$ are vertices of Q and thus belongs to M. If any integer point (z_1, z_2) with $z_1 \geq 0, z_2 \geq 1$ (except $(z_1, z_2) = (0, 1)$) belongs to M, then the point $(0, 1)$ belongs to the interior of the convex hull of (z_1, z_2), $(\frac{29}{28}, \frac{1}{14})$ and $(-\frac{3}{4}, \frac{3}{2})$. Similarly, if an integer point (z_1, z_2) with $z_1 < 0, z_2 \geq 1$ belongs to M, then it satisfies the condition $2z_1 + z_2 \geq 0$ and therefore $(0, 1)$ belongs to the interior of the convex hull of (z_1, z_2), $(0, 0)$ and $(\frac{29}{28}, \frac{1}{14})$. Therefore the only integer point contained in M other than $(0, 0)$ and $(1, 0)$ is $(0, 1)$. Thus, M must be a maximal triangle of type 3 with $(0, 1)$ contained in the relative interior of the third side. ∎

The proof of Corollary 3.4 illustrates that it is not possible to obtain every type 3 triangle cut using a single cross cut (unimodular or otherwise). On the other hand, by Lemma 3.5 all the facet-defining inequalities of W can be obtained using crooked cross cuts. Therefore the crooked cross closure is contained in the cross closure. However, as we do not know if the unimodular and non-unimodular cross closures are obtained by a finite number of inequalities within the respective families, we cannot use Corollary 3.4 and Example 3.9 to conclude that the inclusions in Theorem 3.1 are strict. We believe that these inclusions are strict.

4 Other two row canonical models

The main motivation for studying the set W is that it can be obtained as a relaxation of an MIP by taking two-rows of a simplex tableau (where integer variables are basic) and dropping the (i) integrality of the non-basic variables, and, (ii) nonnegativity of the basic variables. Stronger relaxations of the original set can be obtained by retaining more information about the original set. We next consider two extensions studied in the literature.

4.1 'Monoidal' Strengthening

A stronger two-row relaxation of a mixed-integer set using basic solutions is the following set

$$T = \{(z, s) \in \mathbb{Z}^2 \times \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+ \; : \; z - rs = f\}$$

where the first p nonbasic variables are assumed to be integral. A valid inequality for T obtained via a maximal lattice free convex set B is

$$\sum_{i=1}^{p} \bar{\alpha}_i s_i + \sum_{i=p+1}^{n} \alpha_i s_i \geq 1 \tag{12}$$

where α_i is generated using (2) and $\bar{\alpha}_i = \alpha_i^t$ for some $t \in \mathbb{Z}^{2\times 1}$, where

$$\alpha_i^t = \begin{cases} 0 & \text{if } r_i + t \in \text{rec}(B), \\ 1/\lambda : \lambda > 0 \text{ and } f + \lambda(r_i + t) \in \text{bnd}(B)\} & \text{if } r_i + t \notin \text{rec}(B). \end{cases} \tag{13}$$

The construction of $\bar{\alpha}_i$ is closely related to the fill-in function of Gomory and Johnson [26], and the monoidal strengthening proposed by Balas and Jeroslow [6]. Recently, this construction has been studied in [19], [13], [9].

We next demonstrate that inequality (12) is a crooked cross cut for T. Let $\bar{t}_i \in \mathbb{Z}^2$ for $i = 1, \ldots, p$ be given. Then let $z' = z + \sum_{i=1}^p \bar{t}_i s_i$ and consider the set

$$T' = \{(z', s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ \; : \; z' - r's = f\} \tag{14}$$

where $r'_i = r_i + \bar{t}_i$ for $i \leq p$ and $r'_i = r_i$ for $i \geq p+1$. Now inequality (12) is a 2D lattice-free cut for T' via B. As B is contained in some crooked cross set, inequality (12) is a crooked cross cut for T' and therefore for T. To see this, notice that if (12) is a crooked cross cut for T', then it is valid for the intersections of the four atoms of the associated disjunction with $(T')^{LP}$. Consider the atom defined by $\pi_1 z' \leq \gamma_1$ and $\pi_2 z' \leq \gamma_2$, and note that $\sum_{i=1}^p \bar{\alpha}_i s_i + \sum_{i=p+1}^n \alpha_i s_i \geq 1$ is valid for

$$Q' = \{(z', s) \in \mathbb{R}^{n+2} \; : \; \pi_1 z' \leq \gamma_1, \;\; \pi_2 z' \leq \gamma_2, z' - r's = f, s \geq 0\}.$$

But then, the same inequality is valid for

$$Q = \{(z, s) \in \mathbb{R}^{n+2} \; : \; \pi_1\Big(z + \sum_{i=1}^p \bar{t}_i s_i\Big) \leq \gamma_1, \;\; \pi_2\Big(z + \sum_{i=1}^p \bar{t}_i s_i\Big) \leq \gamma_2, z - rs = f, s \geq 0\}$$

because if the inequality cuts off a point $(z, s) \in Q$, it must also cut off the point $(z', s) \in Q'$ where $z' = z + \sum_{i=1}^p \bar{t}_i s_i$. Repeating the same argument for the remaining three atoms shows that the inequality is indeed a crooked cross cut for T. Consequently, monoidal strengthening simply corresponds to changing the crooked cross disjunction that is used to generate the cut. Similarly, if B is contained in a cross set, then (12) is a cross cut for T.

4.2 S-free cuts

Let N be a rational polyhedron in $\mathbb{R}^2$ and let $S = N \cap \mathbb{Z}^2$. Consider the set

$$\bar{W} := \{(z, s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ \,:\, z - rs = f, z \in S\}, \tag{15}$$

and recall that W can be obtained by dropping the constraint $z \in S$ from $\bar{W}$. We denote the continuous relaxation of $\bar{W}$ obtained by replacing $z \in S$ and $z \in \mathbb{Z}^2$ with $z \in N$ and $z \in \mathbb{R}^2$ by $\bar{W}^{LP}$. Facet-defining inequalities of the convex hull of $\bar{W}$ were first studied by Johnson [28] in the case where S is finite. Recently Dey and Wolsey [21], Fukasawa and Günlük [25], and Basu et al. [10] studied variants of the above set.

Let $B \subseteq \mathbb{R}^2$ be a convex set such that int$(B) \cap S = \emptyset$. Such a set is called an $S-$free convex set. All full-dimensional maximal S-free convex sets are polyhedra [10]. If B is a maximal S-free polyhedron containing f in its interior, then there are row vectors $g_j \in \mathbb{R}^2$ such that $B = \{u \in \mathbb{R}^2 \,:\, g_j(u-f) \leq 1, j \in \{1, ..., k\}\}$. It is shown in [10], [21] and [25] that a valid inequality for $\bar{W}$ (called an S-*free cut*) is $\sum_{i=1}^n \alpha_i s_i \geq 1$, where the coefficients α_i are given by

$$\alpha_i = \max_{1 \leq j \leq k}\{g_j r_i\}. \tag{16}$$

All facet-defining inequalities of (15) that separate the point $(z, s) := (f, 0)$ are S-free cuts obtained from maximal S-free convex sets. In fact, the above papers show the following:

Remark 4.1. *If B is an S-free convex set, then the associated S-free cut is valid for $W^{LP} \setminus (\text{int}(B) \times \mathbb{R}^n)$.*

To see the correctness of the remark above note that any point $(\hat{z}, \hat{s}) \in W^{LP} \setminus (\text{int}(B) \times \mathbb{R}^n)$ satisfies $\hat{z} = f + r\hat{s}$, $\hat{s} \geq 0$ and $\hat{z} \notin \text{int}(B)$. Therefore, $g_l(\hat{z} - f) \geq 1$ for some $l \in \{1, \ldots, k\}$. But then

$$\sum_{i=1}^n \alpha_i \hat{s}_i \geq \sum_{i=1}^n (g_l r_i)\hat{s}_i = g_l \sum_{i=1}^n r_i \hat{s}_i = g_l(\hat{z} - f) \geq 1$$

and therefore $(\hat{z}, \hat{s})$ satisfies the S-free cut. We next show that all S-free cuts are, in fact, crooked-cross cuts.

Theorem 4.2. *Any S-free cut for $\bar{W}$ is a crooked-cross cut for the mixed-integer set obtained by augmenting W with the constraints defining N.*

Proof. Let B be an S-free convex set and $\sum_{i=1}^n \alpha_i s_i \geq 1$ be the associated S-free cut. Let $B' = \text{int}(B) \times \mathbb{R}^n$, and $N' = N \times \mathbb{R}^n$. Now observe that

$$W^{LP} \setminus B' \supseteq \bar{W}^{LP} \setminus B' = \bar{W}^{LP} \setminus (B' \cap N'), \tag{17}$$

where the inclusion follows from the fact that $W^{LP} \supseteq \bar{W}^{LP}$ and the equality follows from the fact that $\bar{W}^{LP} \subseteq N'$. Now using Remark 4.1 and (17), the cut $\sum_{i=1}^n \alpha_i s_i \geq 1$ is also valid for $\bar{W}^{LP} \setminus (B' \cap N')$.

Let F be a maximal lattice-free convex set containing $B \cap N$. Then F is contained in a crooked cross set C. Therefore

$$\bar{W}^{LP} \setminus (B' \cap N') \supseteq \bar{W}^{LP} \setminus (\text{int}(F) \times \mathbb{R}^n) \supseteq \bar{W}^{LP} \setminus (\text{int}(C) \times \mathbb{R}^n).$$

As the S-free cut is valid for the last set in the equation above, and $\bar{W}^{LP}$ equals the points in W^{LP} which satisfy the constraints of N, the Lemma follows from Remark 2.1. ∎

In the proof of Theorem 4.2, if F is also contained in a split or a cross set, then the S-free cut is also, respectively, a split or a cross cut for the mixed-integer set described in Theorem 4.2.

Finally, consider the set $\bar{T} = T \cap \{(z,s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ \ : \ z \in S\}$ where T is as defined in Section 4.1. Given a maximal S-free convex set B, one can use the monoidal strengthening techniques in [6] to generate the cut

$$\sum_{i=1}^{p} \tilde{\alpha}_i s_i + \sum_{i=p+1}^{n} \alpha_i s_i \geq 1 \tag{18}$$

for $\bar{T}$, where $\tilde{\alpha}_i$ equals α_i^t for some $t \in \mathbb{Z}^2 \cap rec(N)$ and

$$\alpha_i^t = \max_{1 \leq j \leq k}\{g_j(r_i + t)\}.$$

Using arguments similar to those in the previous section, we can conclude that the inequality (18) can be obtained using a crooked cross disjunction. Note that in the definition of $\tilde{\alpha}_i$ above, we require t to lie in the recession cone of N. This is because replacing r_i with $r_i + t$ corresponds to replacing z with $z + ts_i$ in the definition of $(\bar{T})'$ in (14). Clearly, $z + ts_i \in S$ provided that $z \in S$ and $t \in rec(N)$.

5 Cross and crooked cross disjunctions for general mixed integer sets

Consider the mixed-integer set with m rows

$$P = \{(x,y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \ : \ Ax + Gy = b, \ y \geq 0\}, \tag{19}$$

where $A \in \mathbb{Q}^{m \times n_1}$, $G \in \mathbb{Q}^{m \times n_2}$ and $b \in \mathbb{Q}^{m \times 1}$. Recall that we use P^{LP} to denote the continuous relaxation of P. Throughout this section, the symbol c denotes a row vector with n_1 components, d, g denote row vectors with n_2 components, and e, f are numbers. The symbols $\alpha, \beta, \gamma, \delta$ denote numbers, $\lambda, \mu, \pi, \tau, \sigma$ are row vectors with $\lambda, \mu \in \mathbb{R}^{1 \times m}$, $\pi \in \mathbb{R}^{1 \times n_1}$ and $\tau \in \mathbb{R}^{1 \times n_2}$. A symbol along with a superscript (e.g., d') has the same dimension as the symbol itself.

We say that an inequality $cx + dy \geq f$ is a translation of $c'x + d'y \geq f'$ w.r.t. P, if there exists a row vector $\mu \in \mathbb{R}^m$ and a positive scalar δ such that $[c, d, f] = \mu[A, G, b] + \delta[c', d', f']$. Given row vectors $\lambda_1, \ldots, \lambda_k \in \mathbb{R}^m$, let

$$P_k(\lambda_1, \ldots, \lambda_k) = \{(x,y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \ : \lambda_i Ax + \lambda_i Gy = \lambda_i b \text{ for } i = 1, \ldots, k, \ y \geq 0\},$$

a particular k-row relaxation of P.

In this section, we show that if a non-trivial (crooked) cross cut for P is obtained from a disjunction defined by row vectors $\pi_1, \pi_2 \in \mathbb{Z}^n$ and $\gamma_1, \gamma_2 \in \mathbb{Z}$, then there exists a three-row relaxation of P of the form

$$P_3(\lambda_1, \lambda_2, \lambda_3) = \{(x,y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \ : \ \pi_1 x + g_1 y = b_1, \ \pi_2 x + g_2 y = b_2, \ g_3 y = b_3, \ y \geq 0\},$$

which yields an equivalent cut under the application of the same (crooked) cross disjunction. When the matrix A has full row rank, $\lambda_3 = 0$ and thus $g_3 = 0$, $b_3 = 0$. Consequently one can obtain the same cut from a two-row relaxation of P; see Corollary 5.9. An important example of a mixed integer set where the matrix A has full row rank is that of a corner relaxation of an MIP. Another special case when a *two-row* relaxation of (19) suffices is presented in Lemma 5.5, whereas Lemma 5.4 and Lemma 5.7 give the precise structure of the three-row relaxation in the general case.

Given a (crooked) cross cut for P, it is easy to obtain a four-row relaxation of P which yields the same cut, *without translation*, under the application of the same (crooked) cross disjunction. Our result implies that there always exist a relaxation with three or fewer rows that also yields an *equivalent* (crooked) cross cut (*via translation*). When the third constraint $g_3y = b_3$ is not needed, for example, when A has full row rank, then the (crooked) cross cut from the relaxation with two rows is in fact a 2D lattice-free cut, obtained by replacing $\pi_1 x$ by z_1 and $\pi_2 x$ by z_2. In general, we expect to obtain stronger cutting planes by applying cross and crooked-cross disjunctions to P instead of generating 2D lattice-free cuts, but we cannot prove this. In contrast, splits cuts can be obtained (up to translations) as MIR cuts (i.e., as 1D lattice-free cuts) from a one-row relaxation of the original MIP [29].

In showing the main result we use the observation that any (crooked) cross cut for P is a split cut for $P \cap \{-\pi_1 x \geq -\gamma_1\}$ and a split cut for $P \cap \{\pi_1 x \geq \gamma_1 + 1\}$. We combine this with the fact that split cuts can be obtained from one row relaxations. The construction of one-row relaxations therefore plays a crucial role in the three-row relaxation we construct for (crooked) cross cuts.

In proving that split cuts are equivalent to MIR cuts and thus showing that split cuts can be obtained from a one row relaxation, one typically has to consider three cases. Assume a split cut is derived from a split disjunction $D_1 \vee D_2$ (here D_1 and D_2 are defined as in Section 2). In the first case, $P^{LP} \cap D_1 \neq \emptyset = P^{LP} \cap D_2$, in which case the only non-trivial split cut is the Gomory-Chvátal cut $\pi x \leq \gamma$, which is also an MIR cut. In the second case, the intersection of P^{LP} with D_1 and D_2 is non-empty, and any split cut can be derived, up to a translation, as an MIR cut. In the final case, $P^{LP} \cap D_1 = P^{LP} \cap D_2 = \emptyset$. Trivially, the split closure is empty. However, both $\pi x \leq \gamma$ and $\pi x \geq \gamma + 1$ are Gomory-Chvátal cuts, and the cut $0x + 0y \geq 1$ is valid for the MIR closure, which is therefore also empty. We discuss the first two cases in Lemma 5.1 in Section 5.1.

In trying to prove an analogous result in Section 5.2 for crooked cross cuts derived from disjunctions of the form $D_1 \vee D_2 \vee D_3 \vee D_4$ (see Section 2), we perform a similar case by case analysis (though the cases will be ordered differently). We consider the case when P^{LP} does not intersect $D_3 \cup D_4$, but intersects $D_1 \cup D_2$ in Lemma 5.7. In Lemma 5.4, we consider the case when P^{LP} intersects both $D_3 \cup D_4$ and $D_1 \cup D_2$. These two results will imply Theorem 5.8. Finally, we consider the case when P^{LP} does not intersect either $D_3 \cup D_4$ or $D_1 \cup D_2$ in Lemma 5.5 and Corollary 5.6. Collectively, these results imply that the (crooked) cross closure always equals the closure with respect to (crooked) cross cuts obtained from three-row relaxations of P (cuts from two-row relaxations suffice when A has full row rank). The first two cases above imply this result when P^{LP} is not contained in the interior of a (crooked) cross set, and the third case implies this result when P^{LP} is contained in the interior of a (crooked) cross set.

5.1 Some results on split cuts

The next property of split cuts can be obtained from the proof of equivalence of split cuts and MIR cuts in [29], or the proof of equivalence of split cuts and GMI cuts in [17]. Even though the result is known, we reprove it in the precise form needed for our main result. Recall the notation introduced in Section 2 for split disjunctions, namely, $D_1 = \{(x, y) \in \mathbb{R}^{n_1+n_2} : \pi x \leq \gamma\}$ and $D_2 = \{(x, y) \in \mathbb{R}^{n_1+n_2} : \pi x \geq \gamma + 1\}$ where $(\pi, \gamma) \in \mathbb{Z}^{n_1+1}$.

Lemma 5.1. *Let $(\pi, \gamma) \in \mathbb{Z}^{n_1+1}$ and assume that $P^{LP} \cap (D_1 \cup D_2) \neq \emptyset$. Let $cx + dy \geq f$ be a nontrivial split cut for P derived from the disjunction $D_1 \vee D_2$, then there exists $\lambda \in \mathbb{R}^{1\times m}$ with $\pi = \lambda A$ such that $\pi x + d'y \geq \gamma + 1$ is a translation of $cx + dy \geq f$ w.r.t. P for some d'. Furthermore, $\pi x + d'y \geq \gamma + 1$ can be derived as a split cut for the one-row relaxation of P,*

$$P_1(\lambda) = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \,:\, \pi x + gy = \beta,\ y \geq 0\},$$

where $g = \lambda G$ and $\beta = \lambda b$, from the same disjunction.

Proof. Let $P^{\leq} = P^{LP} \cap D_1$ and $P^{\geq} = P^{LP} \cap D_2$. By assumption, both $P^{\geq}$ and $P^{\leq}$ cannot be empty and without loss of generality, assume that $P^{\geq}$ is not empty.

If $P^{\leq}$ is empty, then the linear program $z = \min\{\pi x \,:\, x \in P^{LP}\}$ has an optimal value $z^* > \gamma$. Using LP duality, there exist multipliers $\lambda' \in \mathbb{R}^{1\times m}$ such that $\lambda' A = \pi$, $\lambda' G \leq 0$ and $\lambda' b = z^*$. Therefore the implied equation $\lambda'(Ax + Gy = b)$ gives the cut $\pi x \geq \lceil z^* \rceil$, where $\lceil z^* \rceil \geq \gamma + 1$, as an MIR cut. Note that when $P^{\leq}$ is empty, any split cut is dominated by a translation of $\pi x \geq \gamma + 1$. Therefore, in this case $P_1(\lambda')$ gives the desired one-row relaxation.

We now assume that both $P^{\geq}$ and $P^{\leq}$ are nonempty. Let the split cut for P that we consider be $cx + dy \geq f$. In this case, as the cut is valid for both $P^{\geq}$ and $P^{\leq}$, if $cx + dy \geq f$ is not a supporting hyperplane for either $P^{\geq}$ and $P^{\leq}$, then one can trivially get a stronger split cut. We assume that $cx+dy \geq f$ is a supporting hyperplane for $P^{\geq}$. Therefore, there exist multipliers $\lambda_1, \lambda_2 \in \mathbb{R}^{1\times m}$ and $\alpha_1, \alpha_2 \in \mathbb{R}_+$ such that

$$
\begin{aligned}
&c = \lambda_1 A + \alpha_1 \pi, \;\; d \geq \lambda_1 G, \;\; f = \lambda_1 b + \alpha_1(\gamma + 1) \qquad (for \;\; P^{\geq}),\\
&c = \lambda_2 A - \alpha_2 \pi, \;\; d \geq \lambda_2 G, \;\; f \leq \lambda_2 b - \alpha_2(\gamma) \qquad\qquad (for \;\; P^{\leq}).
\end{aligned}
$$

Clearly, $\alpha_1, \alpha_2 > 0$, otherwise the inequality $cx + dy \geq f$ is a trivial inequality valid for P. Observe that

$$
\begin{aligned}
(\lambda_2 - \lambda_1)A &= (\alpha_1 + \alpha_2)\pi,\\
(\lambda_2 - \lambda_1)G &\leq d - \lambda_1 G,\\
(\lambda_2 - \lambda_1)b &\geq (\alpha_1 + \alpha_2)\gamma + \alpha_1.
\end{aligned}
$$

Now consider $P_1((\lambda_2 - \lambda_1)/(\alpha_1 + \alpha_2))$, the relaxation obtained by combining the defining equations of P with the row vector $(\lambda_2 - \lambda_1)/(\alpha_1 + \alpha_2) \in \mathbb{R}^m$. Defining $\tau = d - \lambda_1 G$, it follows that $\tau \geq 0$ and

$$\pi x + \frac{1}{\alpha_1 + \alpha_2}\tau y \geq \gamma + \frac{\alpha_1}{\alpha_1 + \alpha_2} \tag{20}$$

is valid for $P_1((\lambda_2 - \lambda_1)/(\alpha_1 + \alpha_2))$, and therefore so is the MIR cut

$$\frac{\alpha_1}{\alpha_1 + \alpha_2}\pi x + \frac{1}{\alpha_1 + \alpha_2}\tau y \geq (\gamma + 1)\frac{\alpha_1}{\alpha_1 + \alpha_2}. \tag{21}$$

This cut can also be written as $\alpha_1 \pi x + \tau y \geq (\gamma + 1)\alpha_1$; further it is a split cut derived from the same disjunction as $cx + dy \geq f$. Substituting out $\alpha_1 \pi$, τ and $(\gamma + 1)\alpha_1$ gives

$$cx - \lambda_1 Ax + dy - \lambda_1 Gy \geq f - \lambda_1 b.$$

Therefore the inequality (21) divided by $\alpha_1/(\alpha_1 + \alpha_2)$, which has the desired form, is a translation of the original inequality $cx + dy \geq f$. ∎

It is not true in general that a split cut can be obtained as *a split cut without translation* for some one-row relaxation. Moreover, when P^{LP} does not intersect either D_1 or D_2 (and is contained in the interior of the associated split set), Lemma 5.1 does not hold. In this case, not every inequality can be derived up to translation as an MIR cut. For example, let $P = \{x \in \mathbb{Z}^2 \;:\; 0.7 \geq x_1 \geq 0.3\}$ which can be rewritten in our standard form as $\bar{P} = \{(x, y) \in \mathbb{Z}^2 \times \mathbb{R}^2_+ \;:\; x_1 + y_1 = 0.7,\; x_1 - y_2 = 0.3\}$. The cut $x_2 \leq 0$ is a split cut for $\bar{P}$ obtained from the disjunction $(x_1 \leq 0) \vee (x_1 \geq 1)$. Notice that any one-row relaxation of $\bar{P}$ is of the form $(\lambda_1 + \lambda_2)x_1 + \lambda_1 y_1 - \lambda_2 y_2 = 0.7\lambda_1 + 0.3\lambda_2$ and is always feasible for any choice of

x_2 Therefore $x_2 \geq 0$ is not valid for a one-row relaxation; further it cannot be obtained by translation as x_2 does not appear in the constraints defining $\bar{P}$. However, when $P^{LP} \neq \emptyset$ is contained in the interior of a split set, the proof of Lemma 5.1 can be adapted to show that a translation of $0x + 0y \geq 1$ is obtainable from a one-row relaxation.

We next describe a simple class of mixed-integer sets whose convex hull is given by split cuts. We will use this result later in Section 5.2 to distinguish between non-trivial crooked cross cuts and split cuts.

Lemma 5.2. *Consider the mixed integer set* $P := \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}_+ \,:\, Ax + Gy = b\}$*, where* $A \in \mathbb{Q}^{m\times n_1}$*,* $G \in \mathbb{Q}^{m\times n_2}$*, and* $b \in \mathbb{Q}^{m\times 1}$ *and rank*$(A) = 1$*. Then the split closure gives* conv(P)*.*

Proof. As rank$(A) = 1$, we have $A = \eta\sigma$ for some $\eta \in \mathbb{Q}^{m\times 1}$ and $\sigma \in \mathbb{Q}^{1\times n_1}$. Without loss of generality, we assume that $\sigma_i \neq 0$ for $i \in \{1, ..., n_1\}$ since if $\sigma_i = 0$, then the i^{th} column of A is the zero vector and thus x_i can be ignored. Also assume that data defining P, namely, η, σ, G, b are integral and g.c.d.$(\sigma) = 1$. This can be achieved, without loss of generality, by first simultaneously scaling σ, G and b so that σ is integral and g.c.d.$(\sigma) = 1$. Then η, G and b can be scaled simultaneously to make them integral.

Let $Q = \{(z, y) \in \mathbb{Z}^1 \times \mathbb{R}^{n_2}_+ \,:\, \eta z + Gy = b\}$. As Q has a single integer variable, results in [15] imply that the convex hull of Q is given by split cuts. We next considering two cases and show that the polyhedral structure of Q is essentially identical to that of P.

Case 1: Assume $P \neq \emptyset$. We will first show that if $cx + dy \geq f$ is a valid inequality for P, then $c = \delta\sigma$ for some $\delta \in \mathbb{R}$. This claim trivially holds for $n_1 = 1$ so we consider $n_1 \geq 2$. Let $(\hat{x}, \hat{y}) \in P$ and define $x' = (\sigma_2, -\sigma_1, 0, ..., 0)$. As $\sigma x' = 0$, we have $(\hat{x} + kx', \hat{y}) \in P$ for any integer k. Therefore, $c(\hat{x} + kx') + d\hat{y} \geq f$ for any integer k, and consequently, $cx' = 0$, i.e. $c_1\sigma_2 = c_2\sigma_1$. Similarly, it is easy to show that $c_1\sigma_i = c_i\sigma_1$ for all $i \in \{3, ..., n_1\}$. As $\sigma_1 \neq 0$, we obtain $c = \delta\sigma$ where $\delta = c_1/\sigma_1$.

As $(\hat{x}, \hat{y}) \in P$, clearly $(\sigma\hat{x}, \hat{y}) \in Q \neq \emptyset$. This observation implies that if $\delta z + dy \geq f$ is a valid inequality for Q, then $\delta\sigma x + dy \geq f$ is a valid inequality for P. Conversely, notice that for any integer $\hat{z} \in \mathbb{Z}$, there exists $\hat{x} \in \mathbb{Z}^{n_1}$ with $\sigma\hat{x} = \hat{z}$ (as g.c.d.$(\sigma) = 1$). This implies that if $(\hat{z}, \hat{y}) \in Q$ then $(\hat{x}, \hat{y}) \in P$ where $\sigma\hat{x} = \hat{z}$ and, consequently, if $\delta\sigma x + dy \geq \gamma$ is valid for P, then $\delta z + dy \geq \gamma$ is valid for Q.

We have so far established that if $\delta\sigma x + dy \geq f$ is a valid (facet-defining) inequality for conv(P), then $\delta z + dy \geq f$ is valid for Q. As the convex hull of Q is obtained using split cuts, $\delta z + dy \geq f$ is dominated by a nonnegative linear combination of some split cuts $\delta^i z + d^i y \geq f^i$ $(i = 1, ..., t)$ for Q. We will next show that each $\delta^i\sigma x + d^i y \geq f^i$ is a split cut for P for $i = 1, ..., t$.

Suppose $\delta^i z + d^i y \geq f^i$ is a split cut for Q obtained by the disjunction $D_1 \vee D_2$ where $D_1 = \{(z, y) \in \mathbb{R}^1 \times \mathbb{R}^{n_2} \,:\, z \leq \gamma\}$ and $D_2 = \{(z, y) \in \mathbb{R}^1 \times \mathbb{R}^{n_2} \,:\, z \geq \gamma + 1\}$ for some $\gamma \in \mathbb{Z}$. We first consider the case when both $Q^{LP} \cap D_1$ and $Q^{LP} \cap D_2$ are non-empty. In this case, there exists $\lambda_1, \lambda_2 \in \mathbb{R}^{1\times m}$ and $\alpha_1, \alpha_2 \geq 0$ such that

$$\begin{aligned}
&\lambda_1\eta - \alpha_1 = \delta^i, \quad \lambda_1 G \leq d^i, \quad \lambda_1 b - \alpha_1\gamma \geq f^i \\
&\lambda_2\eta + \alpha_2 = \delta^i, \quad \lambda_2 G \leq d^i, \quad \lambda_2 b + \alpha_2(\gamma + 1) \geq f^i
\end{aligned}$$

However, this implies that we can obtain the cut $\delta^i\sigma x + d^i y \geq f^i$ as a split cut for P using the disjunction $\sigma x \leq \gamma \vee \sigma x \geq \gamma + 1$. Next, consider the case $Q^{LP} \cap D_1 = \emptyset$. In this case, it can be verified that the inequality $\delta^i\sigma x + dy \geq \gamma$ is dominated by the cut $\sigma x \geq \eta + 1$ which is a split cut. Similarly when $Q^{LP} \cap D_2 = \emptyset$, it can be verified that the inequality $\delta^i\sigma x + dy \geq f^i$ is dominated by a split cut.

Case 2: Assume $P = \emptyset$. As g.c.d.$(\sigma) = 1$, we also have $Q = \emptyset$. If there is a collection of nontrivial split cuts of the form $\delta^i z + d^i y \geq f^i$ that show that Q is empty set, then $\delta^i\sigma x + d^i y \geq f^i$ are nontrivial split cuts showing that P is empty set. The only remaining case is when there exists a disjunction $D_1 \vee D_2$ such that both $Q^{LP} \cap D_1$ and $Q^{LP} \cap D_1$ are empty. In this case, observe that $P^{LP} \cap \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}_+ : \sigma x \geq \gamma + 1\} = \emptyset$ and $P^{LP} \cap \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}_+ : \sigma x \leq \gamma\} = \emptyset$. Thus the split closure of P is empty. ∎

5.2 Cross and crooked cross cuts for general sets

It is convenient in section to work with *parametric cross cuts* that are a generalization of cross and crooked cross cuts. Let $\pi_1, \pi_2 \in \mathbb{Z}^{1\times n_1}$ and $\gamma_1, \gamma_2 \in \mathbb{Z}$ and consider the *parametric cross* disjunction defined as follows:

$$D_1 = \{(x,y) \in \mathbb{R}^{n_1+n_2} : -\pi_1 x \geq -\gamma_1, \ -(\pi_2 - t\pi_1)x \geq -(\gamma_2 - t\gamma_1)\},$$
$$D_2 = \{(x,y) \in \mathbb{R}^{n_1+n_2} : -\pi_1 x \geq -\gamma_1, \ (\pi_2 - t\pi_1)x \geq \gamma_2 - t\gamma_1 + 1\}$$
$$D_3 = \{(x,y) \in \mathbb{R}^{n_1+n_2} : \ \pi_1 x \geq \gamma_1 + 1, \ -\pi_2 x \geq -\gamma_2\}, \text{ and,}$$
$$D_4 = \{(x,y) \in \mathbb{R}^{n_1+n_2} : \ \pi_1 x \geq \gamma_1 + 1, \ \pi_2 x \geq \gamma_2 + 1\},$$

where t is a non-negative integer. Notice that when $t = 0$ we have a cross disjunction and when $t = 1$ we have a crooked cross disjunction. Let $Q_i = P^{LP} \cap D_i$ for $i = 1, \ldots, 4$.

We will need the following basic property of parametric cross cuts.

Lemma 5.3. *If $cx + dy \geq f$ is a non-trivial parametric cross cut for P derived from the disjunction $\vee_{i=1}^4 D_i$, then π_1 and π_2 are linearly independent.*

Proof. Assume that π_1 and π_2 are linearly dependent. If $\pi_2 = 0$, then for any t one can verify that $\cup_{i=1}^4 D_i = \{(x,y) : \pi_1 x \leq \gamma_1\} \cup \{(x,y) : \pi_1 x \geq \gamma_1 + 1\}$. Therefore the parametric cross disjunction is just a split disjunction and $cx + dy \geq f$ is a split cut. Similarly, $cx + dy \geq f$ is trivially a split cut for P when $\pi_1 = 0$. We can thus assume that $\pi_1 \neq 0$ and $\pi_2 = \delta\pi_1$ for some $\delta \neq 0$.

We first assume that g.c.d.(π_1) = g.c.d.$(\pi_2) = 1$. Next observe that because of the restriction on the g.c.d. of π_1 and π_2, $\delta = \pm 1$ and is thus integral. The atoms $D_1, \ldots, D_4$ become, respectively,

$$\{(x,y) : \ \pi_1 x \leq \gamma_1, \ (\delta - t)\pi_1 x \leq \gamma_2 - t\gamma_1\}, \{(x,y) : \ \pi_1 x \leq \gamma_1, \ (\delta - t)\pi_1 x \geq \gamma_2 - t\gamma_1 + 1\},$$

$$\{(x,y) : \ \pi_1 x \geq \gamma_1 + 1, \ \delta\pi_1 x \leq \gamma_2\}, \text{ and } \{(x,y) : \ \pi_1 x \geq \gamma_1 + 1, \ \delta\pi_1 x \geq \gamma_2 + 1\}.$$

Now, the set of points $\{(x,y) : \pi_1 x \in \mathbb{Z}\}$ is clearly contained in $\cup_{i=1}^4 D_i$. After all, if $\bar{x} \in \mathbb{R}^{n_1}$ and $\pi_1\bar{x}$ is integral, then either $\pi_1\bar{x} \leq \gamma_1$ or $\pi_1\bar{x} \geq \gamma_1 + 1$. In the first case, as $(\delta - t)$ is integral, so is $(\delta - t)\pi_1\bar{x}$ and therefore this number cannot lie strictly between $\gamma_2 - t\gamma_1$ and $\gamma_2 - t\gamma_1 + 1$. One argues similarly in the second case. Therefore, $cx + dy \geq f$ is valid for $P^{LP} \cap \{(x,y) : \pi_1 x \in \mathbb{Z}\}$. However, Cook, Kannan and Schrijver [15] showed that the split closure of P is contained in

$$conv(P^{LP} \cap \{(x,y) : \pi_1 x \in \mathbb{Z}\}). \tag{22}$$

Therefore, $cx + dy \geq f$ is valid for the split closure of P, a contradiction. Therefore, the claim holds when g.c.d.$(\pi_1) = 1$ and g.c.d.$(\pi_2) = 1$.

In the general case, as π_1 and π_2 are integral, there must exist an integer vector π' and co-prime integers r, s such that $\pi_1 = s\pi', \pi_2 = r\pi'$ and $\delta = r/s$. One can then show that $cx + dy \geq f$ is valid for (22) with π_1 replaced by π', and consequently π_1 and π_2 have to be linearly independent for any non-trivial parametric cross cut. ∎

We next prove a series of lemmas which differ primarily in the assumptions on which atoms have a non-empty intersection with P^{LP}. A basic idea that we use in the proofs is the observation that for a given parametric cross cut, the cut is a split cut for $P^{LP} \cap \{-\pi_1 x \geq -\gamma_1\}$ using the disjunction

$$(\pi_2 - t\pi_1)x \leq (\gamma_2 - t\gamma_1) \ \vee \ (\pi_2 - t\pi_1)x \geq \gamma_2 - t\gamma_1 + 1,$$

and it is a split cut for $P^{LP} \cap \{\pi_1 x \geq \gamma_1 + 1\}$ using the disjunction $\pi_2 x \leq \gamma_2 \ \vee \ \pi_2 x \geq \gamma_2 + 1$. In the next lemma, we assume that both $Q_1 \cup Q_2$ and $Q_3 \cup Q_4$ are non empty. In this case, using the fact that translations of split cuts can be derived from one-row relaxations, we can argue that any parametric cross cut can be generated from a three-row relaxation of the following form.

Lemma 5.4. *Let $cx + dy \geq f$ be a non-trivial parametric cross cut for P derived from the disjunction $\vee_{i=1}^4 D_i$. If $Q_1 \cup Q_2 \neq \emptyset$ and $Q_3 \cup Q_4 \neq \emptyset$, then there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^{1\times m}$ with $\pi_i = \lambda_i A$ for $i = 1, 2$ and $\lambda_3 A = 0$ such that a translation of $cx + dy \geq f$ is a parametric cross cut for the three-row relaxation of P,*

$$P_3(\lambda_1, \lambda_2, \lambda_3) = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \ : \ \pi_1 x + g_1 y = b_1, \ \pi_2 x + g_2 y = b_2, \ g_3 y = b_3, \ y \geq 0\},$$

where $g_i = \lambda_i G$, $b_i = \lambda_i b$ for $i = 1, 2, 3$, derived from the disjunction $\vee_{i=1}^4 D_i$.

Proof. As $cx + dy \geq f$ is valid for both Q_3 and Q_4, it is a split cut for

$$P^{\geq} = \{(x, y, s) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2+1} \ : \ Ax + Gy = b, \ \pi_1 x - s_1 = \gamma_1 + 1, \ s_1, y \geq 0\}$$

derived from the disjunction $(\pi_2 x \leq \gamma_2) \vee (\pi_2 x \geq \gamma_2 + 1)$. As $Q_3 \cup Q_4$ is not empty, by Lemma 5.1 there exists a translation of $cx + dy \geq f$ for $P^{\geq}$ which has the form $\pi_2 x + d'y + e's_1 \geq f'$ and can be derived as a split cut for the following one-row relaxation of $P^{\geq}$:

$$P_1^{\geq}([\lambda_1, \alpha_1]) = \left\{(x, y, s) \in \mathbb{Z}^{n_1} \times \mathbb{R}_+^{n_2+1} \ : \ (\lambda_1 A + \alpha_1 \pi_1)x + \lambda_1 G y - \alpha_1 s_1 = \lambda_1 b + \alpha_1(\gamma_1 + 1)\right\},$$

where $\lambda_1 \in \mathbb{R}^m$, $\alpha_1 \in \mathbb{R}$ and we have $\pi_2 = \lambda_1 A + \alpha_1 \pi_1$. We will refer to the inequality $cx + dy \geq f$ as the *original* cut and the inequality $\pi_2 x + d'y + e's_1 \geq f'$ as *cut (A)*.

As cut (A) is a translation of the original cut for the set $P^{\geq}$ we have

$$\beta_1 \begin{bmatrix} c \\ d \\ 0 \\ f \end{bmatrix} = \begin{bmatrix} \pi_2 \\ d' \\ e' \\ f' \end{bmatrix} + \begin{bmatrix} \mu_1 A \\ \mu_1 G \\ 0 \\ \mu_1 b \end{bmatrix} + \delta_1 \begin{bmatrix} \pi_1 \\ 0 \\ -1 \\ \gamma_1 + 1 \end{bmatrix} \tag{23}$$

for some $\beta_1, \delta_1 \in \mathbb{R}^1$ and $\mu_1 \in \mathbb{R}^{1\times m}$ where $\beta_1 > 0$. Clearly, $\delta_1 = e'$. As cut (A) is a split cut for $P_1^{\geq}([\lambda_1, \alpha_1])$, it is also a split cut for the two-row set

$$T = \{(x, y, s) \in \mathbb{Z}^{n_1} \times \mathbb{R}_+^{n_2+1} \ : \ \lambda_1 A x + \lambda_1 G y = \lambda_1 b, \ \pi_1 x - s_1 = \gamma_1 + 1\}.$$

Notice that $\pi_1 x - s_1 = \gamma_1 + 1$ is included in the definition of this set and therefore adding a multiple of this equation to cut (A), we can obtain the following cut

$$(\pi_2 + \delta_1 \pi_1)x + d'y \geq f' + \delta_1(\gamma_1 + 1), \tag{24}$$

which we will call *cut (B)*, as a split cut for T. By rewriting (23) note that coefficients of cut (B) satisfy

$$\begin{bmatrix} \pi_2 + \delta_1 \pi_1 \\ d' \\ 0 \\ f' + \delta_1(\gamma_1 + 1) \end{bmatrix} = \beta_1 \begin{bmatrix} c \\ d \\ 0 \\ f \end{bmatrix} - \begin{bmatrix} \mu_1 A \\ \mu_1 G \\ 0 \\ \mu_1 b \end{bmatrix}$$

and therefore cut (B) is a translation of the original cut for P. We will work with cut (B) in the rest of the proof and for notational ease write it as $\bar{c}x+\bar{d}y \geq \bar{f}$, where $\bar{c} = (\pi_2+\delta_1\pi_1)$, $\bar{d} = d'$ and $\bar{f} = f'+\delta_1(\gamma_1+1)$.

So far, we showed that cut (B) is a translation of the original cut for P and it can be obtained as a split cut for T derived from the disjunctions $(\pi_2 x \leq \gamma_2) \vee (\pi_2 x \geq \gamma_2 + 1)$. Therefore, cut (B) is valid for $T^{LP} \cap \{x : \pi_2 x \leq \gamma_2\} = T^{LP} \cap D_3 \supseteq P_1^{LP}(\lambda_1) \cap D_3$ and, similarly, also for $P_1^{LP}(\lambda_1) \cap D_4$.

We now look at the remaining atoms associated with the parametric cross disjunction. As the original cut is valid for Q_1 and Q_2, it is a split cut for

$$P^{\leq} = \{(x,y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \;:\; Ax + Gy = b,\; \pi_1 x + s_2 = \gamma_1,\; s_2, y \geq 0\}$$

derived from the disjunction $((\pi_2 - t\pi_1)x \geq \gamma_2 - t\gamma_1 + 1) \vee ((\pi_2 - t\pi_1)x \leq \gamma_2 - t\gamma_1)$. As cut (B) is a translation of the original cut, cut (B) also is a split cut for $P^{\leq}$ derived from the same disjunction. Therefore, as $Q_1 \cup Q_2$ is not empty, there exists a translation of cut (B) for $P^{\leq}$ which has the form $(\pi_2 - t\pi_1)x + d''y + e''s_2 \geq f''$, which we will call cut (C). Clearly cut (C) can be derived as a split cut for the following one-row relaxation of $P^{\leq}$:

$$P_1^{\leq}([\lambda_2, \alpha_2]) = \left\{(x,y) \in \mathbb{Z}^{n_1} \times \mathbb{R}_+^{n_2} \;:\; (\lambda_2 A + \alpha_2\pi_1)x + \lambda_2 G y + \alpha_2 s_2 = \lambda_2 b + \alpha_2\gamma_1\right\},$$

where $\lambda_2 \in \mathbb{R}^m$, $\alpha_2 \in \mathbb{R}$ and we have $\pi_2 - t\pi_1 = \lambda_2 A + \alpha_2\pi_1$. As before, it is easy to see that cut (C) is a split cut for the two-row set

$$K = \{(x,y,s) \in \mathbb{Z}^{n_1} \times \mathbb{R}_+^{n_2+1} \;:\; \lambda_2 Ax + \lambda_2 Gy = \lambda_2 b,\; \pi_1 x + s_2 = \gamma_1\}.$$

Furthermore, as cut (C) is a translation of cut (B) for $P^{\leq}$,

$$\beta_2 \begin{bmatrix} \bar{c} \\ \bar{d} \\ 0 \\ \bar{f} \end{bmatrix} = \begin{bmatrix} \pi_2 - t\pi_1 \\ d'' \\ e'' \\ f'' \end{bmatrix} + \begin{bmatrix} \mu_2 A \\ \mu_2 G \\ 0 \\ \mu_2 b \end{bmatrix} + \delta_2 \begin{bmatrix} \pi_1 \\ 0 \\ 1 \\ \gamma_1 \end{bmatrix} \tag{25}$$

for some $\beta_2, \delta_2 \in \mathbb{R}^1$ and $\mu_2 \in \mathbb{R}^{1\times m}$ where $\beta_2 > 0$ and $\delta_2 = -e''$. Notice that cut (B) may not be a split cut for K. However, cut (B) is a split cut for the three-row set

$$L = \{(x,y,s) \in \mathbb{Z}^{n_1} \times \mathbb{R}_+^{n_2+1} \;:\; \lambda_2 Ax + \lambda_2 Gy = \lambda_2 b,\; \mu_2 Ax + \mu_2 Gy = \mu_2 b,\; \pi_1 x + s_2 = \gamma_1\},$$

and therefore cut (B) is valid for $P_2^{LP}(\lambda_2, \mu_2) \cap D_1$ and $P_2^{LP}(\lambda_2, \mu_2) \cap D_2$.

Now consider the three-row relaxation of P given by the multipliers λ_1, λ_2 and $\lambda_3 = \mu_2$:

$$P_3(\lambda_1, \lambda_2, \lambda_3) = \{(x,y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \;:\; \lambda_i Ax + \lambda_i Gy = \lambda_i b,\; i = 1,2,3\,; y \geq 0\},$$

and note that $P_3(\lambda_1, \lambda_2, \lambda_3) = P_1^{LP}(\lambda_1) \cap P_2^{LP}(\lambda_2, \mu_2)$. Clearly, cut (B) is valid for $P_3(\lambda_1, \lambda_2, \lambda_3) \cap D_i$ for $i = 1, \ldots, 4$ and therefore it is a parametric cross cut for $P_3(\lambda_1, \lambda_2, \lambda_3)$. We have therefore established that a translation of the original cut, namely cut (B), can be obtained as a parametric cross cut from a three-row relaxation of P. We will next show that the coefficients of the x variables in this three-row relaxation satisfy the desired properties.

Remember that π_1 and π_2 are assumed to be linearly independent, and

$$M = \begin{bmatrix} \lambda_1 A \\ \lambda_2 A \\ \lambda_3 A \end{bmatrix} = \begin{bmatrix} \pi_2 - \alpha_1\pi_1 \\ \pi_2 - t\pi_1 - \alpha_2\pi_1 \\ (\beta_2 - 1)\pi_2 + (\beta_2\delta_1 + t - \delta_2)\pi_1 \end{bmatrix} = \begin{bmatrix} -\alpha_1 & 1 \\ -(t+\alpha_2) & 1 \\ (\beta_2\delta_1 + t - \delta_2) & (\beta_2 - 1) \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}.$$

Clearly the matrix M has rank either 1 or 2. If rank$(M) = 2$, then there exists a non-singular 3×3 matrix M' such that $M'M = [\pi_1, \pi_2, 0]^T$ and the relaxation $P_3(\lambda_1, \lambda_2, \lambda_3)$ can indeed be written in the desired form by multiplying the defining equations by M'.

On the other hand, if rank$(M) = 1$, then by Lemma 5.2 the convex hull of $P_3(\lambda_1, \lambda_2, \lambda_3)$ can be obtained using split cuts. However, this implies that the inequality $cx + dy \geq f$ is valid for the split closure of P as $P_3(\lambda_1, \lambda_2, \lambda_3)$ is obtained by taking linear combinations of the rows of P. Since we assumed that the original cut $cx + dy \geq f$ is a nontrivial cross cut, and therefore so is its translation $\bar{c}x + \bar{d}y \geq \bar{f}$, we obtain that rank$(M) = 2$, completing the proof. ∎

We would like to emphasize that both cut (B) and cut (C) in the proof above are translations of the original cut with respect to P. We cannot, however show that either of them is a parametric cross cut for $P_2(\lambda_1, \lambda_2)$ when we do not have the translation row $g_3 y = b_3$.

Lemma 5.5. *Let $\vee_{i=1}^4 D_i$ be a parametric cross disjunction, and assume $Q_3 \cup Q_4 = \emptyset$ (resp., $Q_1 \cup Q_2 = \emptyset$). Then $\pi_1 x \leq \gamma_1$ (resp., $\pi_1 x \geq \gamma_1 + 1$) is a parametric cross cut for P. If it is a non-trivial one, then there exist $\lambda_1, \lambda_2 \in \mathbb{R}^{1\times m}$ with $\pi_1 = \lambda_1 A, \pi_2 = \lambda_2 A$ such that it is a parametric cross cut for*

$$P_2(\lambda_1, \lambda_2) = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \ : \ \pi_1 x + g_1 y = b_1, \ \pi_2 x + g_2 y = b_2, \ y \geq 0\},$$

where $g_i = \lambda_i G$ and $b_i = \lambda_i b$ for $i = 1, 2$, derived from $\vee_{i=1}^4 D_i$.

Proof. By definition, $\pi_1 x \leq \gamma_1$ is valid for D_1 and D_2. Also, $\pi_1 x \leq \gamma_1$ is valid for $\emptyset = Q_3 = Q_4$, and is thus a parametric cross cut for P.

Now assume $\pi_1 x \leq \gamma_1$ is a non-trivial parametric cross cut, i.e., it is not valid for the split closure of P. Therefore, the split closure of P (and also P^{LP}) cannot equal $\emptyset$. As $Q_3 = Q_4 = \emptyset$, the Farkas' Lemma implies that there exist multipliers $\lambda_1', \lambda_2' \in \mathbb{R}^m$, and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}_+$ such that

$$\lambda_1' A + \alpha_1 \pi_1 - \beta_1 \pi_2 = 0, \quad \lambda_1' G \leq 0, \quad \lambda_1' b + \alpha_1(\gamma_1 + 1) - \beta_1 \gamma_2 \geq 1, \tag{26}$$

$$\lambda_2' A + \alpha_2 \pi_1 + \beta_2 \pi_2 = 0, \quad \lambda_2' G \leq 0, \quad \lambda_2' b + \alpha_2(\gamma_1 + 1) + \beta_2(\gamma_2 + 1) \geq 1. \tag{27}$$

As $\pi_1 x \leq \gamma_1$ is a non-trivial parametric cross cut, Lemma 5.3 implies that π_1 and π_2 are linearly independent. If $\lambda_1' = 0$, the linear independence of π_1 and π_2 would imply that $\alpha_1 = \beta_1 = 0$ which would contradict the last inequality in (26). Therefore $\lambda_1' \neq 0$, and we can similarly conclude that $\lambda_2' \neq 0$. Therefore, $P_2(\lambda_1', \lambda_2')$ is a non-trivial relaxation of P, and $P_2^{LP}(\lambda_1', \lambda_2') \cap D_3 = \emptyset$ and $P_2^{LP}(\lambda_1', \lambda_2') \cap D_4 = \emptyset$. This implies that $\pi_1 x \leq \gamma_1$ is a cross cut for $P_2(\lambda_1', \lambda_2')$.

We can assume that at least one of α_1, β_1 is positive, otherwise, $P^{LP} = \emptyset$, a contradiction to the assumed non-triviality of $\pi_1 x \leq \gamma_1$. Similarly, we conclude that one of α_2, β_2 is positive. If $\alpha_1 = 0$, then $\beta_1 > 0$; this would imply that P^{LP} has an empty intersection with $\{x : \pi_2 x \leq \gamma_2\}$. Then points in P^{LP} would satisfy $\pi_2 x > \gamma_2$. Similarly, if $\alpha_2 = 0$, then points in P^{LP} would satisfy $\pi_2 x < \gamma_2 + 1$. Thus if both α_1 and α_2 are zero, then both the inequalities $\pi_2 x \geq \gamma_2 + 1$ and $\pi_2 x \leq \gamma_2$ are split cuts, and the split closure of P is empty, a contradiction. Therefore at least one of α_1, α_2 is positive.

If $\beta_1 = 0$ or $\beta_2 = 0$, then $\pi_1 x < \gamma_1 + 1$ is a valid inequality for P^{LP}, and therefore $\pi_1 x \leq \gamma_1$ is a split cut for P. As we assumed that $\pi_1 x \leq \gamma_1$ is not a split cut for P, we can assume that $\beta_1 > 0$ and $\beta_2 > 0$. Now, equations (26) and (27) imply that

$$\begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix} A = \Omega \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} \text{ where } \Omega = \begin{pmatrix} -\alpha_1 & \beta_1 \\ -\alpha_2 & -\beta_2 \end{pmatrix}$$

The determinant of Ω equals $\alpha_1\beta_2 + \alpha_2\beta_1$ which is positive, and therefore Ω is invertible. Letting

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \Omega^{-1} \begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix},$$

we see that $\lambda_1 A = \pi_1$ and $\lambda_2 A = \pi_2$. Further, the relaxation $P_2(\lambda_1, \lambda_2) = P_2(\lambda_1', \lambda_2')$, as the constraints of either relaxation can be obtained from the constraints of the other by linear combinations.

Notice that if the intersection of P^{LP} with D_1 and D_2 is empty, then arguing as above, we would infer that $\pi_1 x \geq \gamma_1 + 1$ is a parametric cross cut for the two-row relaxation $P_2(\lambda_1, \lambda_2)$; further the coefficients of the x variables in the first constraint of $P_2(\lambda_1, \lambda_2)$ would be π_1 and in the second constraint would be $\pi_2 - t\pi_1$. Clearly, we could then conclude that $P_2(\lambda_1, \lambda_2) = P_2(\lambda_1, \lambda_2 + t\lambda_1)$, and that the second relaxation has the desired form. The result follows. ∎

Corollary 5.6. *Let the intersection of P^{LP} with every atom of a parametric cross disjunction $\vee_{i=1}^4 D_i$ be empty. Then $0x + 0y \geq 1$ is a linear combination of parametric cross cuts. Therefore the parametric cross closure of P equals $\emptyset$.*

Proof. If P^{LP} satisfies the conditions of the corollary, then Lemma 5.5 implies that $\pi_1 x \leq \gamma_1$ (or $-\pi_1 x \geq -\gamma_1$) and $\pi_1 x \geq \gamma_1 + 1$ are parametric cross cuts for P. Adding these cuts together, we obtain $0x + 0y \geq 1$. ∎

Lemma 5.7. *Let $cx + dy \geq f$ be a non-trivial parametric cross cut for P derived from the disjunction $\vee_{i=1}^4 D_i$, and assume $Q_3 = Q_4 = \emptyset$, but $Q_1 \cup Q_2 \neq \emptyset$. Then there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^{1\times m}$ with $\pi_i = \lambda_i A$ for $i = 1, 2$ and $\lambda_3 A = 0$ such that a translation of $cx + dy \geq f$ is a parametric cross cut for $P_3(\lambda_1, \lambda_2, \lambda_3)$.*

Proof. Let λ_1, λ_2 be defined as in Lemma 5.5. Then the LP relaxation of the two-row relaxation $P_2(\lambda_1, \lambda_2)$ has empty intersection with D_3 and D_4. Further, as in the proof of Lemma 5.4, we can obtain $cx + dy \geq f$ as a split cut for $P^{\leq}$, and thereby from a one-row relaxation of $P^{\leq}$. From this, obtain the multiplier λ' (= λ_2 in Lemma 5.4) such that $cx + dy \geq f$ is valid for $P_1^{LP}(\lambda') \cap D_1$ and $P_1^{LP}(\lambda') \cap D_2$. Then $cx + dy \geq f$ is valid for the intersection of each atom of the disjunction D with $P_3^{LP}(\lambda_1, \lambda_2, \lambda')$. Finally, from the proof of Lemma 5.4 it is clear that $\lambda' A$ is a linear combination of π_1 and π_2; more precisely in the notation of Lemma 5.4, $\lambda' A = \pi_2 - (t + \alpha_2)\pi_1$. Then setting $\lambda_3 = \lambda' - \lambda_2 + (t + \alpha_2)\lambda_1$, we see that $P_3(\lambda_1, \lambda_2, \lambda') = P_3(\lambda_1, \lambda_2, \lambda_3)$, and the latter relaxation has the desired form. ∎

Together Lemma 5.4 and Lemma 5.7 imply the following result.

Theorem 5.8. *Let $cx + dy \geq f$ be a non-trivial parametric cross cut for P derived from the disjunction $\vee_{i=1}^4 D_i$. If $P^{LP} \cap (\cup_{i=1}^4 D_i) \neq \emptyset$, then there exist row vectors $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^m$ with $\pi_i = \lambda_i A$ for $i = 1, 2$ and $\lambda_3 A = 0$ such that a translation of $cx + dy \geq f$ is a parametric cross cut for the three-row relaxation of P, $P_3(\lambda_1, \lambda_2, \lambda_3)$, derived from the disjunction $\vee_{i=1}^4 D_i$.*

This result can be viewed as a generalization of Lemma 5.1, which is a statement about the equivalence of split cuts and MIR cuts.

5.3 A special case

Consider the case when the coefficient matrix A has full row rank and note that in this case $\lambda A = 0$ implies that $\lambda = 0$ and therefore we obtain the following as a corollary of Theorem 5.8.

Corollary 5.9. *Let $cx + dy \geq f$ and P satisfy the conditions of Theorem 5.8. If A has full row rank, then there exist row vectors $\lambda_1, \lambda_2 \in \mathbb{R}^m$ with $\pi_i = \lambda_i A$ for $i = 1, 2$ such that a translation of $cx + dy \geq f$ is a parametric cross cut for the two-row relaxation of P, $P_2(\lambda_1, \lambda_2)$, derived from the disjunction $\vee_{i=1}^{4} D_i$.*

Consider a non-trivial cross cut $cx + dy \geq f$ for a set P where A has full row rank. By Corollary 5.9, $cx + dy \geq f$ is a cross cut for $P_2(\lambda_1, \lambda_2)$. Consider any atom of the disjunction which has nonempty intersection with $P_2^{LP}(\lambda_1, \lambda_2)$, and obtain multipliers for its constraints and for the defining inequalities of the atom, say $\pi_1 x \leq \gamma_1$ and $\pi_2 x \leq \gamma_2$, which prove that $cx + dy \geq f$ is a valid inequality. Then c is a (uniquely defined) linear combination of π_1 and π_2. Therefore, we can subtract multiples of the constraints of $P_2(\lambda_1, \lambda_2)$ to obtain an equivalent cut $d'y \geq f'$, which is a cross cut for the two-row relaxation. But replacing $\pi_1 x$ by z_1 and $\pi_2 x$ by z_2, we see that $d'y \geq f'$ is a unimodular cross cut for a set having the same form as W in (1), and is thus implied by a quadrilateral cut or a triangle cut of type 1 or 2. Thus, if A has full row rank, then there is a 1-to-1 correspondence (up to translations) between non-trivial cross cuts and the family of quadrilateral cuts for P and triangle cuts of type 1 or 2. Similarly, there is a 1-to-1 correspondence (up to translations) between crooked cross cuts and 2D lattice-free cuts for P.

In particular, consider the following MIP

$$z \; = \; min\{cx + dy \; : \; Ax + Gy = b, x \in \mathbb{Z}^{n_1}, \; y \in \mathbb{R}^{n_2}, \; x \geq 0, \; y \geq 0\}$$

where all data is rational. Let a basis for the LP relaxation of this problem be given and using this basis, the variables are rewritten as $x = [x_B, x_N]$ and $y = [y_B, y_N]$ where x_B, y_B denote the basic variables and x_N, y_N denote the nonbasic variables. Similarly, the constraint matrix is rewritten as $[A_B, A_N, G_B, G_N]$. We call the following relaxation of the MIP, where non-negativity requirements of the nonbasic variables are dropped, its *corner relaxation*:

$$z_C \; = \; min\{cx + dy \; : \; Ax + Gy = b, x \in \mathbb{Z}^{n_1}, \; y \in \mathbb{R}^{n_2}, \; x_N \geq 0, \; y_N \geq 0\}.$$

This relaxation is a natural extension of the well-known corner relaxation of pure integer linear programs (IP) introduced by Ralph Gomory (see [27], [14] for recent discussions). Clearly, the corner relaxation, both for IPs and MIPs, depends on the choice of the LP basis.

Note that this feasible region of this corner relaxation can equivalently be written as

$$\begin{aligned}
x_B + A^1 x_N + G^1 y_N &= b^1 && (28)\\
y_B + A^2 x_N + G^2 y_N &= b^2 && (29)\\
x_N - Is &= 0 && (30)\\
s, y_N &\geq 0 && (31)\\
x_B, x_N &\in \mathbb{Z} && (32)
\end{aligned}$$

after (i) writing the nonnegativity of x_N variables using additional slack variables, and, (ii) multiplying the original constraint matrix with the inverse of the (basis) matrix obtained by collecting the columns associated with the basic variables.

Notice that y_B variables are not restricted in sign and given x_B, x_N, y_N that satisfy (28), (30)-(32), it is always possible to set $y_B = b^2 - A^2 x_N - G^2 y_N$ to obtain a feasible point. Therefore, dropping variables y_B and equations (29) simply corresponds to projecting out the y_B variables and does not change the value of the relaxation. Now note that the coefficients of the integer variables in the remaining constraints form the matrix

$$\bar{A} \; = \; \left[\begin{array}{cc} I_B & A^1 \\ 0 & I_N \end{array} \right]$$

where I_B and I_N are identity matrices of appropriate dimension. Clearly $\bar{A}$ has full-row rank and therefore satisfies the conditions of Corollary 5.9. Therefore the family of crooked cross (cross) cuts for a corner relaxation of an MIP equals the family of 2D lattice-free cuts (quadrilateral cuts and triangle cuts of type 1 and 2).

6 Concluding remarks

One of our main contributions in this paper is to define the crooked cross cuts and show that, for general mixed-integer sets, they dominate 2D lattice-free cuts. Even though we believe this dominance to be strict, we are not able to establish it. In other words, we are not able to answer if a two-row relaxation is sufficient to produce any given (crooked) cross cut. Depending on the answer to this question, one can conclude if crooked cross cuts strictly dominate 2D lattice-free cuts or not. We think this is a very interesting open problem.

Another related question we have not looked at in this paper is whether or not the (crooked) cross cut closure of a polyhedral mixed-integer set is polyhedral. Note that polyhedrality of the cross cut closure would not immediately imply the polyhedrality of the quadrilateral closure of the canonical two-row set. This is due to another open question regarding the dominance of cross cuts over unimodular cross cuts.

A final related question that we find interesting is whether or not crooked cross cuts give the convex hull of mixed-integer sets with only two integer variables. If true, this would be an extension of the fact that split cuts give the convex hull of mixed-integer sets with a single integer variable. We believe the answer to be affirmative.

Recently there has been much research into deriving effective cutting planes for MIPs using 2D lattice-free cuts. One computational approach which has been explored is to obtain a two-row continuous group relaxation from two simplex tableau rows corresponding to basic integer variables with fractional value, derive 2D lattice free cuts, and then apply lifting to incorporate upper bound information and integrality of variables (see [8], [23], [24]). It seems hard to devise an effective algorithm to find good, maximal, lattice-free bodies such that the associated cuts are violated by the current LP solution. The results in this paper suggest an alternative to generating 2D lattice free cuts for MIPs. Rather than creating a two-row continuous group relaxation from a pair of simplex tableau rows, one can directly apply different crooked cross disjunctions to these rows, though it is probably hard to choose a good crooked cross disjunction. Answers to such computational questions require extensive experimentation.

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